Student's Solutions Manual

for

Differential Equations:
Theory, Technique, and Practice
with Boundary Value Problems

Second Edition

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Chapter 1

What is a Differential Equation?

1.1 Introductory Remarks

1.2 A Taste of Ordinary Differential Equations

1.3 The Nature of Solutions

1. Verify the function is a solution to the differential equation.

(a) If $y = x^2 + c$, then $y' = 2x$.

(b) If $y = cx^2$, then $y' = 2cx$ so $xy' = 2cx^2 = 2y$.

(c) If $y^2 = e^{2x} + c$, then $2yy' = 2e^{2x}$ so $yy' = e^{2x}$.

(d) If $y = ce^{kx}$, then $y' = kce^{kx}$ so $y' = ky$.

(e) If $y = c_1 \sin 2x + c_2 \cos 2x$, then $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$ and $y'' = -4c_1 \sin 2x - 4c_2 \cos 2x = -4y$ so $y'' + 4y = 0$.

(f) If $y = c_1 e^{2x} + c_2 e^{-2x}$, then $y' = 2c_1 e^{2x} - 2c_2 e^{-2x}$ and $y'' = 4c_1 e^{2x} + 4c_2 e^{-2x} = 4y$ so $y'' - 4y = 0$.

(g) If $y = c_1 \sinh 2x + c_2 \cosh 2x$, then $y' = 2c_1 \cosh 2x + 2c_2 \sinh 2x$ and $y'' = 4c_1 \sinh 2x + 4c_2 \cosh 2x = 4y$ so $y'' - 4y = 0$.

(h) If $y = \arcsin xy$, then $y' = \frac{xy'+y}{\sqrt{1-(xy)^2}}$ so $xy' + y = y'\sqrt{1 - x^2y^2}$.

(i) If $y = x\tan x$, then $y' = x\sec^2 x + \tan x = x(\tan^2 x + 1) + \tan x$. Using $\tan x = y/x$ we get $y' = y^2/x + x + y/x$ or $xy' = x^2 + y^2 + y$.

(j) If $x^2 = 2y^2 \ln y$, then $2x = [2y^2(1/y) + 4y\ln y]y' = 2yy'(1 + 2\ln y)$. Consequently, $y' = \frac{x}{y+2y\ln y}$. Using $\ln y = \frac{x^2}{2y^2}$ we get $y' = \frac{xy}{x^2+y^2}$.

(k) If $y^2 = x^2 - cx$, then $2yy' = 2x - c$ so $2xyy' = 2x^2 - cx = x^2 + x^2 - cx = x^2 + y^2$.

(l) If $y = c^2 + c/x$, then $y' = -c/x^2$ so $x^4(y')^2 = c^2 = y - c/x$. Use the fact that $-c/x = xy'$ to obtain $x^4(y')^2 = y + xy'$.

(m) If $y = ce^{y/x}$, then $y' = \frac{xy'-y}{x^2}ce^{y/x} = \frac{xyy'-y^2}{x^2}$. Solve for y' to obtain $y' = y^2/(xy - y^2)$.

(n) If $y+\sin y = x$, then $y'+y'\cos y = 1$ or $y' = 1/(1+\cos y)$. Multiply the numerator and denominator of the right side by y to obtain $y' = y/(y + y\cos y)$. Now use the identity $y = x - \sin y$ to obtain $y' = (x - \sin y + y\cos y)$.

(o) If $x + y = \arctan y$, then $1 + y' = y'/(1 + y^2)$. Consequently, $(1 + y')(1 + y^2) = y'$. This simplifies to $1 + y^2 + y^2y' = 0$.

3. For each of the following differential equations, find the particular solution that satisfies the given initial condition.

(a) If $y' = xe^x$, then $y = \int xe^x dx + C = (x - 1)e^x + C$ (integrate by parts, $u = x$). When $x = 1$, $y = C$ so the particular solution is $y(x) = (x - 1)e^x + 3$.

(b) If $y' = 2\sin x\cos x$, then $y = \int 2\sin x\cos x dx + C = \sin^2 x + C$. When $x = 0$, $y = C$ so the particular solution is $y(x) = \sin^2 x + 1$.

(c) If $y' = \ln x$, then $y = \int \ln x dx + C = x\ln x - x + C$ (integrate by parts, $u = \ln x$). When $x = e$, $y = C$ so the particular solution is $y(x) = x\ln x - x$.

(d) If $y' = 1/(x^2 - 1)$, then $y = \int 1/(x^2 - 1)dx + C = 1/2\int 1/(x - 1) - 1/(x+1)dx + C = \frac{1}{2}\ln\frac{x-1}{x+1} + C$ (method of partial fractions). When $x = 2$, $y = \frac{1}{2}\ln\frac{1}{3} + C = C - \frac{\ln 3}{2}$ so the particular solution is $y(x) = \frac{1}{2}\ln\frac{x-1}{x+1} + \frac{\ln 3}{2}$.

(e) If $y' = \frac{1}{x(x^2-4)}$, then $y = \int \frac{1}{x(x^2-4)} dx + C = 1/8 \int 1/(x+2) + 1/(x-2) - 2/x dx + C = \frac{1}{8} \ln \frac{|x^2-4|}{x^2} + C$ (method of partial fractions). When $x = 1$, $y = \frac{1}{8} \ln 3 + C$ so the particular solution is $y(x) = \frac{1}{8} \ln \frac{|x^2-4|}{x^2} - \frac{1}{8} \ln 3 = \frac{1}{8} \ln \frac{|x^2-4|}{3x^2}$.

(f) If $y' = \frac{2x^2+x}{(x+1)(x^2+1)}$, then $y = \int \frac{2x^2+x}{(x+1)(x^2+1)} dx + C = \frac{1}{2} \int \frac{1}{x+1} + \frac{3x-1}{x^2+1} dx + C = \frac{1}{2} \ln(x+1) + \frac{3}{4} \ln(x^2+1) - \frac{1}{2} \arctan x + C$ (method of partial fractions). When $x = 0$, $y = C$ so the particular solution is $y(x) = \frac{1}{2} \ln(x+1) + \frac{3}{4} \ln(x^2+1) - \frac{1}{2} \arctan x + 1$.

5. For the differential equation

$$y'' - 5y' + 4y = 0,$$

carry out the detailed calculations required to verify these assertions.

(a) If $y = e^x$, then $y'' - 5y' + 4y = e^x - 5e^x + 4e^x \equiv 0$.
If $y = e^{4x}$, then $y'' - 5y' + 4y = 16e^{4x} - 20e^{4x} + 4e^{4x} \equiv 0$.

(b) If $y = c_1 e^x + c_2 e^{4x}$, then $y'' - 5y' + 4y = c_1(e^x - 5e^x + 4e^x) + c_2(16e^{4x} - 20e^{4x} + 4e^{4x} \equiv 0$.

7. For which values of m will the function $y = y_m = e^{mx}$ be a solution of the differential equation

$$2y''' + y'' - 5y' + 2y = 0?$$

Find three such values m. Use the ideas in Exercise 5 to find a solution containing three arbitrary constants c_1, c_2, c_3.

Substitute $y = e^{mx}$ into the differential equation to obtain

$$2m^3 e^{mx} + m^2 e^{mx} - 5me^{mx} + 2e^{mx} = 0.$$

Cancel e^{mx} in each term (it is never 0) to obtain the equivalent equation

$$2m^3 + m^2 - 5m + 2 = 0.$$

Observing that $m = m_1 = 1$ is a solution (and $y_1 = e^x$ is a solution to the differential equation). Using this we can factor the polynomial–divide by $m - 1$–to obtain

$$2m^3 + m^2 - 5m + 2 = (m - 1)(2m^2 + 3m - 2).$$

The quadratic term factors yield two more roots, $m_2 = -2$, $m_3 = 1/2$, and two more solutions

$$y_2 = e^{-2x} \text{ and } y_3 = e^{x/2}.$$

These three solutions can be combined, as in Exercise 5, to produce a solution with three arbitrary constants

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^{x/2}.$$

1. Use the method of separation of variables to solve each of these ordinary differential equations.

(a) Write the equation $x^5 y' + y^5 = 0$ in Leibnitz form $x^5 \frac{dy}{dx} + y^5 = 0$ and separate the variables: $\frac{dy}{y^5} = -\frac{dx}{x^5}$. Integrate, $\int \frac{dy}{y^5} = -\int \frac{dx}{x^5}$, to obtain the solution: $y^{-4}/(-4) = x^{-4}/4 + C$. This can also be written in the form $x^4 + y^4 = Cx^4y^4$ or $y = (\frac{x^4}{Cx^4-1})^{1/4}$.

(b) Write the equation $y' = 4xy$ in Leibnitz form $\frac{dy}{dx} = 4xy$ and separate the variables: $\frac{dy}{y} = 4xdx$. Integrate, $\int \frac{dy}{y} = \int 4xdx$, to obtain the solution: $\ln|y| = 2x^2 + C$. This can also be written in the form $y = Ce^{2x^2}$.

(c) Write the equation $y' + y\tan x = 0$ in Leibnitz form $\frac{dy}{dx} + y\tan x = 0$ and separate the variables: $\frac{dy}{y} = -\tan x dx$. Integrate, $\int \frac{dy}{y} == \int \tan x dx$, to obtain the solution: $\ln|y| = \ln|\cos x| + C$. This can also be written in the form $y = C\cos x$.

(d) The equation $(1+x^2)dy + (1+y^2)dx = 0$ can be rearranged and integrated directly, $\int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = C$. Therefore, the implicit solution is $\arctan y + \arctan x = C$. This can also be written in the form $y = \tan(C - \arctan x)$.

(e) Proceed as in part (d). Rearrange $y \ln y dx - xdy = 0$ to the form $\frac{dx}{x} - \frac{dy}{y\ln y} = 0$ and integrate: $\int \frac{dx}{x} - \int \frac{dy}{y \ln y} = C$. This yields the implicit solution $\ln|x| - \ln|\ln y| = C$ which can also be written in the form $\ln y = Cx$ or $y = e^{Cx}$.

(f) From Leibnitz form $x\frac{dy}{dx} = (1-4x^2)\tan y$ we obtain $\cot y dy = (1/x - 4x)dx$. Integrating, $\int \cot y dy = \int 1/x - 4xdx$, gives the implicit solution $\ln|\sin y| = \ln|x| - 2x^2 + C$. Consequently, $\sin y = Cxe^{-2x^2}$ so $y = \arcsin(Cxe^{-2x^2})$.

(g) The Leibnitz form $\frac{dy}{dx}\sin y = x^2$ separates to $\sin y dy = x^2 dx$. Integrate to obtain $-\cos y = x^3/3 + C$ or $y = \arccos(C - x^3/3)$.

(h) Write the equation $y' - y\tan x = 0$ in Leibnitz form $\frac{dy}{dx} - y\tan x = 0$ and separate the variables: $\frac{dy}{y} = \tan x dx$. Integrate $\int \frac{dy}{y} = \int \tan x dx$, to obtain the solution: $\ln|y| = -\ln|\cos x| + C$. This can also be written in the form $y = C/\cos x$ or $y = C\sec x$.

(i) From Leibnitz form, $xy\frac{dy}{dx} = y - 1$ we obtain $\frac{ydy}{y-1} = \frac{dx}{x}$. Write this in the form $\frac{y-1+1}{y-1}dy = \frac{dx}{x}$ and integrate to obtain the implicit solution $y + \ln|y-1| = \ln|x| + C$.

(j) Leibnitz form $xy^2 - \frac{dy}{dx}x^2 = 0$ separates to $\frac{dx}{x} = y-2dy$. Integrating yields the implicit solution $\ln|x| = -1/y + C$. The solution can be expressed explicitly as $y = \frac{1}{C-\ln|x|}$.

3. Substituting $y' = p$ yields $\frac{p'}{p} = x^2$. Separation of variables (or direct integration) gives $\ln|p| = x^3/3 + C$. This implies that $p = Ce^{x^3/3}$ and so $y' = Ce^{x^3/3}$. Consequently, $y = C\int e^{x^3/3}dx + D$. As we expect, the solution contains two arbitrary constants. The integral cannot be evaluated in terms of elementary functions.

1.4 First-Order Linear Equations

1. Find the general solution of the following equations.

(a) The equation is linear and separable. The integrating factor is $e^{-x^2/2}$ so it simplifies to $(e^{-x^2/2}y)' = 0$ and $e^{-x^2/2}y = C$. Therefore, $y = Ce^{x^2/2}$.

(b) This equation is also linear and separable. The integrating factor is $e^{x^2/2}$ so it simplifies to $(e^{x^2/2y}y)' = xe^{x^2/2}$. Integrate to obtain $e^{x^2/2}y = e^{x^2/2} + C$ and $y = 1 + Ce^{-x^2/2}$.

(c) The integrating factor is e^x so the equation simplifies to $(e^x y)' = \frac{e^x}{1+e^{2x}}$. Integrating we obtain $e^x y = \arctan e^x + C$. The general solution is $y = e^{-x}\arctan(e^x) + Ce^{-x}$.

(d) The integrating factor is e^x so the equation simplifies to $(e^x y)' = 2x + x^2 e^x$. Integrate to get $e^x y = x^2 + (2-2x+x^2)e^x + C$ (integrate

x^2e^x by parts, twice, or use an integral table). The general solution is $y = x^2e^{-x} + 2 - 2x + x^2 + Ce^{-x}$.

(e) Write as $xy' = 2y - x^3$ and then $y' - (2/x)y = -x^2$. The integrating factor is x^{-2} so the equation simplifies to $(x^{-2}y)' = -1$. Integrate to obtain $x^{-2}y = -x + C$. The general solution is $y = -x^3 + Cx^2$.

(f) The integrating factor is e^{x^2} so the equation simplifies to $(e^{x^2}y)' = 0$. Consequently, $e^{x^2}y = C$ and $y = Ce^{-x^2}$.

(g) Write as $y' - (3/x)y = x^3$. The integrating factor is x^{-3} so the equation simplifies to $(x^{-3}y)' = 1$. Integrating we obtain $x^{-3}y = x + C$ so $y = x^4 + Cx^3$.

(h) Express the equation in the form $y' + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}$. The integrating factor is $1 + x^2$ and the equation simplifies to $((1 + x^2)y)' = \cot x$. Consequently, $(1 + x^2)y = \ln|\sin x| + C$ and $y = \frac{\ln|\sin x|+C}{1+x^2}$.

(i) The integrating factor is $\sin x$ and the equation simplifies to $(y \sin x)' = 2x$. Therefore, $y \sin x = x^2 + C$ and $y = \frac{x^2+C}{\sin x}$.

(j) Express the equation in the form $y' + (1/x + \cot x)y = 1$. The integrating factor is $x \sin x$ so the equation simplifies to $(xy \sin x)' = x \sin x$. Integrate to obtain $xy \sin x = \sin x - x \cos x + C$ (use a table of integrals or integrate $x \sin x$ by parts, $u = x$). Therefore, $y = \frac{\sin x - x\cos x + C}{x \sin x}$.

3. Bernoulli Equations To verify the technique write the Bernoulli equation in the form $y^{-n}y' + Py^{1-n} = Q$. The substitution $z = y^{1-n}$ and $z' = (1 - n)y^{-n}y'$ yield $\frac{1}{1-n}z' + Pz = Q$.

(a) Bernoulli, $n = 3$. Substitute $z = y^{-2}$, $z' = -2y^{-3}y'$ into $xy^{-3}y' + y^{-2} = x^4$ to obtain $(-1/2)xz' + z = x^4$. This is linear, $z' - (2/x)z = -2x^3$, with integrating factor x^{-2}. It simplifies to $(x^{-2}z)' = -2x$. COnsequently, $x^{-2}z = -x^2 + C$ and $z = -x^4 + Cx^2$. This means that $y^{-2} = Cx^2 - x^4$ so $y^2 = \frac{1}{Cx^2-x^4}$.

(b) Write the equation in the form $y' + (1/x)y = y^{-2}\cos x$ to see that it is Bernoulli, $n = -2$. Substitute $z = y^3$, $z' = 3y^2y'$ into the equation $y^2y' + (1/x)y^3 = \cos x$ to obtain the linear equation $(1/3)z' + (1/x)z = \cos x$. This is $z' + (3/x)z = 3\cos x$, with integrating factor x^3. It simplifies to $(x^3z)' = 3x^3 \cos x$. Consequently, $x^3z = 3F(x) + C$ where $F(x)$ is and antiderivative for $x^3 \cos x$.

Therefore, $z = 3x^{-3}F(x) + x^{-3}C$, and the solution in terms of y can be expressed in the form $y^3x^3 = 3F(x) + C$. The expression $F(x)$ can be found using multiple integration by parts or, better yet, a table of integrals: $F(x) = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x$.

5. If $y = cf(x) + g(x)$, then $g' = cf'(x) + g'(x)$. Estimate the constant c (multiply the first equation by $f'(x)$, the second by $f(x)$, and subtract) to obtain $f'(x)y - f(x)y' = f(x)g(x) - f'(x)g'(x)$. This is a first-order linear equation.

7. The equation is linear. Write it in the form $y' - (2 \csc 2x)y = \frac{2\cos x}{\sin 2x}$ to find the integrating factor: $\csc 2x + \cot 2x = \cot x$. Since $\frac{2\cos x}{\sin 2x} = \frac{2\cos x}{2 \sin x \cos x} = \csc x$, upon multiplying by the integrating factor we obtain $(y \cot x)' = \csc x \cot x$. Therefore, $y \cot x = -\csc x + C$ and $y = \frac{-\csc x + C}{\cot x}$. Converting to sines and cosines yields $y = \frac{C \sin x - 1}{\cos x}$. As $x \to \frac{\pi}{2}$ the denominator approaches 0. To get a finite limit the numerator must also approach 0 so let $C = 1$. The solution $y = \frac{\sin x - 1}{\cos x}$ has the desired property.

9. Let $x(t)$ be the amount of salt (pounds) in the tank at time t (minutes). The rate of change, $x'(t)$, equals the rate in: 6 lbs/min, minus the rate out: $3 \cdot \frac{x(t)}{V(t)}$ lbs/min, where $V(t) = 40 - t$ is the volume of the mixture in the tank at time t. The resulting rate equation: $x' = 6 - 3 \cdot \frac{x}{40-t}$, is linear. Write it as $x' + \frac{3}{40-t}x = 6$ to obtain the integrating factor, $(40-t)^{-3}$. The equation simplifies to $[(40-t)^{-3}x]' = 6(40-t)^{-3}$ implying that $(40-t)^{-3}x = 3(40-t)^{-2} + C$. Consequently, $x(t) = 3(40-t) + C(40-t)^3$. The initial condition, $x(0) = 0$, determines the value of the constant: $C = -\frac{3}{1600}$. Therefore, the amount of salt in the tank at time t is given by

$$x(t) = 3(40-t) - \frac{3}{1600}(40-t)^3.$$

(a) Since $V(t) = 40 - t$ there are 20 gallons of brine in the tank when $t = 20$. According to the formula derived above, $x(20) = 45$ pounds of salt.

(b) The amount of salt in the tank is maximum when $x'(t) = 0$. Solving

$$-3 + \frac{9}{1600}(40-t)^2 = 0$$

we first obtain $(40-t)^2 = \frac{1600}{3}$, then $t = 40 - \frac{40}{\sqrt{3}} = 16.905\ldots$.

1.5 Exact Equations

All references to M and N refer to the equation in the form $Mdx + Ndy = 0$.

1. $M = y$, $N = x + \frac{2}{y}$, so $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = y$ implies that $f(x,y) = xy + \phi(y)$. $\frac{\partial f}{\partial y} = x + \frac{2}{y}$ implies that $x + \phi'(y) = x + \frac{2}{y}$ so $\phi(y) = 2\ln|y|$ and $f(x,y) = xy + 2\ln|y|$. The implicit solution is $xy + 2\ln|y| = C$.

3. $M = y - x^3$, $N = x + y^3$, so $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = y - x^3$ implies that $f(x,y) = xy - x^4/4 + \phi(y)$. $\frac{\partial f}{\partial y} = x + y^3$ implies that $x + \phi'(y) = x + y^3$ so $\phi(y) = y^4/4$ and $f(x,y) = xy - x^4/4 + y^4/4$. The implicit solution is $xy - x^4/4 + y^4/4 = C$.

5. $M = y + y\cos xy$, $N = x + x\cos xy$, so $\frac{\partial M}{\partial y} = 1 - xy\sin xy + \cos xy = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = y + y\cos xy$ implies that $f(x,y) = xy + \sin xy + \phi(y)$. $\frac{\partial f}{\partial y} = x + x\cos xy$ implies that $x + x\cos xy + \phi'(y) = x + x\cos xy$, so $\phi(y) = 0$ and $f(x,y) = xy + \sin xy$. The implicit solution is $xy + \sin xy = C$. For a given value of C, the equation $t + \sin t = C$ has exactly one solution, t_0. Therefore, the implicit solution for the differential equation also be expressed simply as $xy = C$.

7. $M = e^y + \cos x\cos y$, $N = xe^y - \sin x\sin y$, so $\frac{\partial M}{\partial y} = e^y - \cos x\sin y = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = e^y + \cos x\cos y$ implies that $f(x,y) = xe^y + \sin x\cos y + \phi(y)$. $\frac{\partial f}{\partial y} = xe^y - \sin x\sin y$ implies that $xe^y - \sin x\sin y + \phi'(y) = xe^y - \sin x\sin y$ so $\phi(y) = 0$ and $f(x,y) = xe^y + \sin x\cos y$. The implicit solution is $xe^y + \sin x\cos y = C$.

9. $M = 1 + y$, $N = 1 - x$, so $\frac{\partial M}{\partial y} = 1 \neq -1 = \frac{\partial N}{\partial x}$. The equation is not exact.

11. $M = \frac{y}{1-x^2y^2} - 1$, $N = \frac{x}{1-x^2y^2}$ so $\frac{\partial M}{\partial y} = \frac{1+x^2y^2}{(1-x^2y^2)^2} = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = \frac{y}{1-x^2y^2} - 1$ implies that

$$f(x,y) = \frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) - x + \phi(y).$$

$\frac{\partial f}{\partial y} = \frac{x}{1-x^2y^2}$ implies that $\frac{1}{2}(\frac{x}{xy+1} - \frac{x}{xy-1}) + \phi'(y) = \frac{x}{1-x^2y^2}$. This simplifies to $\frac{x}{1-x^2y^2} + \phi'(y) = \frac{x}{1-x^2y^2}$, so $\phi(y) = 0$ and $f(x,y) = \frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) - x$. The implicit solution is $\frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) - x = C$. This equation can be solved for y (exercise) to produce an explicit solution $y = \frac{Ce^{2x}+1}{x(Ce^{2x}-1)}$.

13. $M = \frac{y}{1-x^2y^2} + x$, $N = \frac{x}{1-x^2y^2}$, so $\frac{\partial M}{\partial y} = \frac{1+x^2y^2}{(1-x^2y^2)^2} = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = \frac{y}{1-x^2y^2} + x$ implies that

$$f(x,y) = \frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) + \frac{x^2}{2} + \phi(y).$$

$\frac{\partial f}{\partial y} = \frac{x}{1-x^2y^2}$ implies that $\frac{1}{2}(\frac{x}{xy+1} - \frac{x}{xy-1} + \phi'(y) = \frac{x}{1-x^2y^2}$. This simplifies to $\frac{x}{1-x^2y^2} + \phi'(y) = \frac{x}{1-x^2y^2}$, so $\phi(y) = 0$ and $f(x,y) = \frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) + \frac{x^2}{2}$. The implicit solution is $\frac{1}{2}\ln(xy+1) - \frac{1}{2}\ln(xy-1) + \frac{x^2}{2} = C$. This equation can be solved for y (exercise) to produce an explicit solution $y = \frac{Ce^{-x^2}+1}{x(Ce^{-x^2}-1)}$.

15. $M = x\ln y + xy$, $N = y\ln x + xy$, so $\frac{\partial M}{\partial y} = \frac{x}{y} + x \neq \frac{y}{x} + y = \frac{\partial N}{\partial x}$. The equation is not exact.

17. $M = 1 + y^2\sin 2x$, $N = -2y\cos^2 x$, so $\frac{\partial M}{\partial y} = 2y\sin 2x = 4y\cos x\sin x = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = 1 + y^2\sin 2x$ implies that $f(x,y) = x - \frac{1}{2}y^2\cos 2x + \phi(y)$. $\frac{\partial f}{\partial y} = -2y\cos^2 x$ implies that $-y\cos 2x + \phi'(y) = -2y\cos^2 x$. Since $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, the last equation is equivalent to $-y\cos 2x + \phi'(y) = -y - y\cos 2x$ so $\phi'(y) = -y$, $\phi(y) = -\frac{1}{2}y^2$, and $f(x,y) = x - \frac{1}{2}y^2\cos 2x - \frac{1}{2}y^2$. Therefore, the implicit function is $x - \frac{1}{2}y^2\cos 2x - \frac{1}{2}y^2 = C$.

19. $M = 3x^2(1+\ln y)$, $N = \frac{x^3}{y} - 2y$ so $\frac{\partial M}{\partial y} = \frac{3x^2}{y} = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = 3x^2(1+\ln y)$ implies that $f(x,y) = x^3(1+\ln y) + \phi(y)$. $\frac{\partial f}{\partial y} = \frac{x^3}{y} - 2y$ implies that $\frac{x^3}{y} + \phi'(y) = \frac{x^3}{y} - 2y$, so $\phi'(y) = -2y$, $\phi(y) = -y^2$, and $f(x,y) = x^3(1+\ln y) - y^2$. The implicit solution is $x^3(1+\ln y) - y^2 = C$.

(a) $\frac{\partial f}{\partial x} = \frac{y}{(x+y)^2} - 1$ implies that $f(x,y) = -\frac{y}{x+y} - x + \phi(y)$. $\frac{\partial f}{\partial y} = 1 - \frac{x}{(x+y)^2}$ implies that $1 - \frac{x}{(x+y)^2} = phi'(y) - \frac{x}{(x+y)^2}$ so $\phi'(y) = 1$,

$\phi(y) = y$ and $f(x,y) = -\frac{y}{x+y} - x + y$. The implicit solution is $-\frac{y}{x+y} - x + y = C$.

(b) $\frac{\partial f}{\partial y} = 1 - \frac{x}{(x+y)^2}$ implies that $f(x,y) = y + \frac{x}{x+y} + \varphi(x)$. $\frac{\partial f}{\partial x} = \frac{y}{(x+y)^2} - 1$ implies that $\frac{y}{(x+y)^2} + \varphi'(x) = \frac{y}{(x+y)^2} - 1$ so $\varphi'(x) = -1$, $\varphi(x) = -x$ and $f(x,y) = y + \frac{x}{x+y} - x$. The implicit solution is $y + \frac{x}{x+y} - x = D$.

Subtracting the left side of the solutions from (a) and (b) yields $(-\frac{y}{x+y} - x + y) - (y + \frac{x}{x+y} - x) = -1$. Differing by a constant -1, two equations from (a) and (b) represent the same family of equations.

21. $M = \frac{4y^2-2x^2}{4xy^2-x^3}$, $N = \frac{8y^2-x^2}{4y^3-x^2y}$, so $\frac{\partial M}{\partial y} = \frac{8xy}{(x^2-4y^2)^2} = \frac{\partial N}{\partial x}$. The equation is exact. $\frac{\partial f}{\partial x} = \frac{4y^2-2x^2}{4xy^2-x^3}$ implies that $f(x,y) = \ln x + \frac{1}{2}\ln(x^2-4y^2) + \phi(y)$ (integrate via partial fractions). $\frac{\partial f}{\partial y} = \frac{8y^2-x^2}{4y^3-x^2y}$ implies that $\frac{4y}{x^2-4y^2} + \phi'(y) = \frac{8y^2-x^2}{4y^3-x^2y}$, so $\phi'(y) = \frac{1}{y}$. Therefore, $\phi(y) = \ln y$, and $f(x,y) = \ln x + \frac{1}{2}\ln(x^2-4y^2) + \ln y$. The solution is $\ln xy + \frac{1}{2}\ln(x^2-4y^2) = C$ or $x^2y^2(x^2-4y^2) = C$.

1.6 Orthogonal Trajectories and Families of Curves

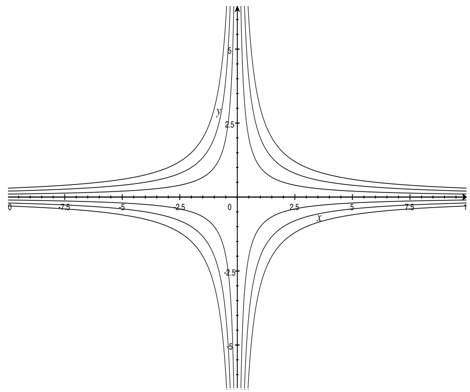
1. Sketch the families of curves, find orthogonal trajectories, add them to the sketch.

(a) Differentiate $xy = c$ to get $xy' + y = 0$ or $y' = -y/x$. Orthogonal curves are generated by $y' = x/y$, a separable equation. $ydy = xdx$ implies that $y^2/2 = x^2/2 + C$ or $y^2 - x^2 = C$. See Figure **??**.

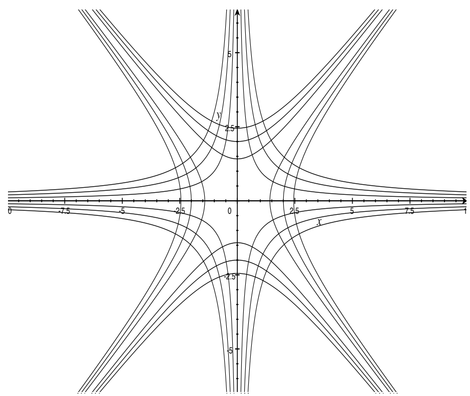
(b) Differentiate $y = cx^2$ to get $y' = 2cx$. Divide one by the other, $y'/y = 2/x$ or $y' = 2y/x$. Orthogonal curves are generated by $y' = -\frac{x}{2y}$, a separable equation. $2ydy = -xdx$ implies that $y^2 = -x^2/2 + C$ or $2y^2 + x^2 = C$. See **??**.

(c) Differentiate $x + y = c$ to get $1 + y' = 0$ or $y' = -1$. Orthogonal curves are generated by $y' = 1$ or $y = x + C$. See Figure **??**.

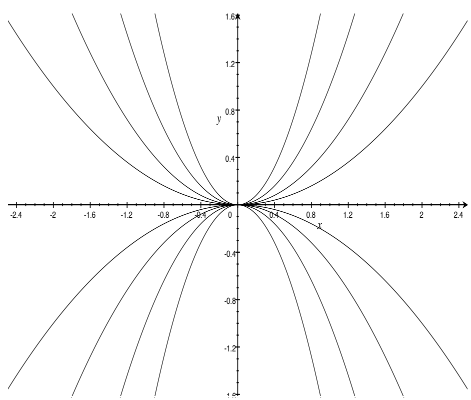
(d) The curves $r = c(1 + \cos\theta)$ for c positive are orthogonal to the curves $r = c(1 + \cos\theta)$ for c negative. See Figure **??** where the

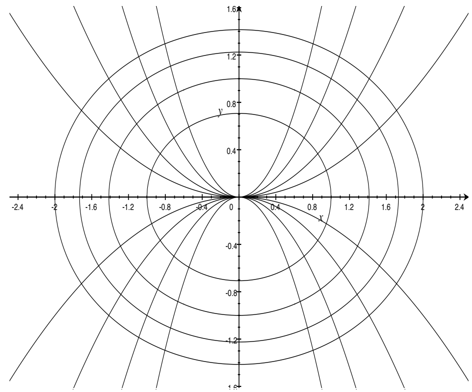


The family $xy = c$ and orthogonal trajectories.

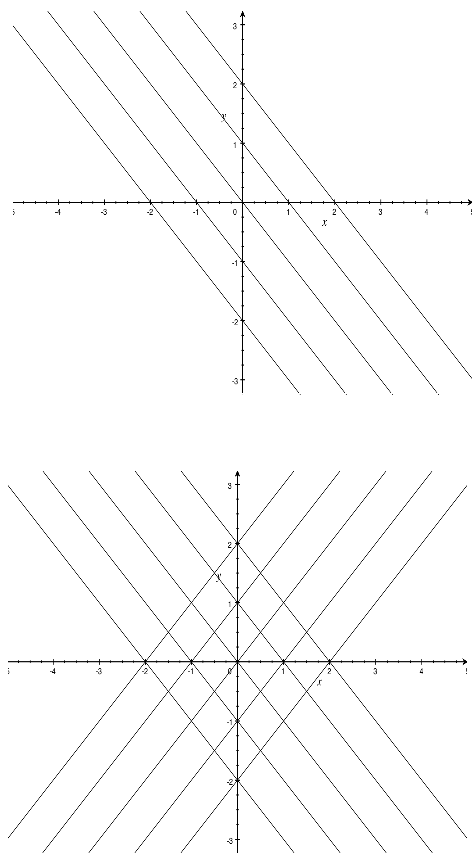


More on the family $xy = c$ and orthogonal trajectories.

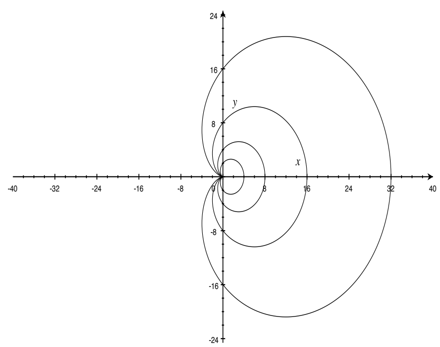




The family $y = cx^2$ and orthogonal trajectories.



The family $x + y = c$ and orthogonal trajectories.



orthogonal curves on the right correspond to $c = -2, -4, -8, -16$. To verify this analytically, write the whole family (c positive and negative) in Cartesian coordinates: $\pm\sqrt{x^2 + y^2} = c(1 + \frac{x}{\pm\sqrt{x^2+y^2}})$, and differentiate. Then use the original equation to eliminate c to obtain the following differential equations:

$$x + yy' = \frac{y^2 - xyy'}{x \pm \sqrt{x^2 + y^2}}.$$

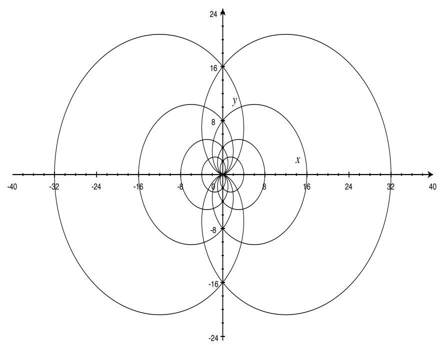
The differential equations for the orthogonal trajectories can now be obtained by replacing y' with $-\frac{1}{y'}$. Having done this, multiply both sides of the equation by yy', then "rationalize" the denominator on the right (multiply top and bottom by $x \mp \sqrt{x^2 + y^2}$) and rearrange to

$$\frac{y^2 - xyy'}{x \mp \sqrt{x^2 + y^2}} = x + yy'.$$

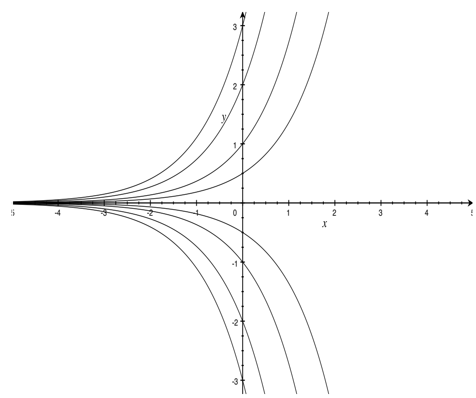
Because these are the same equations, the family is "self-orthogonal".

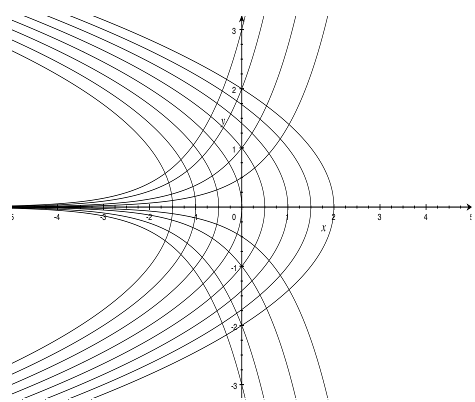
(e) Differentiate $y = ce^x$ to get $y' = ce^x$. Divide one by the other to obtain $y'/y = 1$ or $y' = y$. Orthogonal curves are generated by the separable equation $y' = -1/y$ or $ydy = -dx$. Integrate to obtain $y^2/2 = -x + C$. This simplifies to $y^2 + 2x = C$. See Figure **??**.

(f) Differentiate $x - y^2 = c$ to get $1 - 2yy' = 0$ or $y' = 1/(2y)$. Orthogonal curves are generated by the separable equation $y' = -2y$ or $dy/y = -2dx$. INtegrate to obtain $\ln|y| = -2x + C$. This simplifies to $y = Ce^{-2x}$. See Figure **??**.

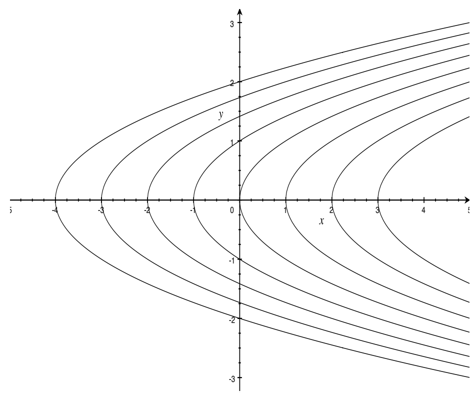


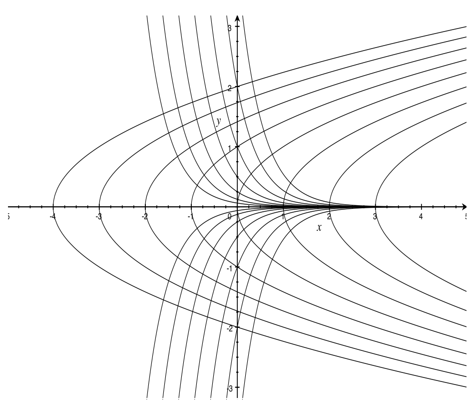
The family $r = c(1 + \cos\theta), c = 2, 4, 8, 16$ and orthogonal trajectories.



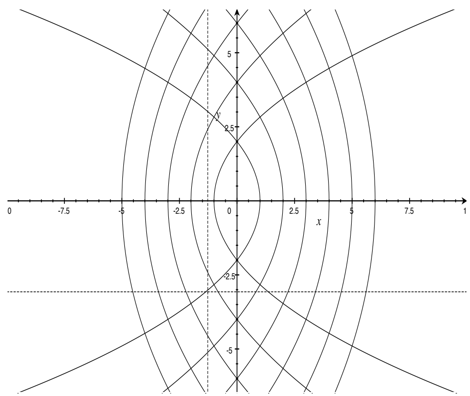


The family $y = ce^x$ and orthogonal trajectories.





The family $y = cx^4$ and orthogonal trajectories.



The family $x - y^2 = c$ and orthogonal trajectories.

3. The sketch is displayed in Figure **??**. It appears that the curves intersect orthogonally.

Differentiate $y^2 = 4c(x+c)$ to get $2yy' = 4c$. Substitute $c = yy'/2$ into the first equation to obtain $y^2 = 2yy'(x + yy'/2)$ or $y = 2xy' + y(y')^2$. This is the differential equation defining the parabolas. Replacing y' with $-1/y'$ yields the differential equation defining the orthogonal trajectories: $y = -\frac{2x}{y'} + y \cdot \frac{1}{(y')^2}$. It simplifies to $y(y')^2 = -2xy' + y$ which is equivalent to the original, confirming the fact that the parabolas in Figure **??** are orthogonal to one another.

5. Let (x, y) be a point on the curve. The area from 0 to x is $\int_0^x y(t)dt$. The area of the rectangle is $x \cdot y(x)$ so $\int_0^x y(t)dt = \frac{x \cdot y(x)}{3}$. Differentiate with respect to x to obtain $y(x) = \frac{xy'(x)+y(x)}{3}$ or $3y = xy' + y$. This is a separable equation $\frac{dy}{y} = \frac{2dx}{x}$. Integrate to obtain $\ln y = 2\ln x + C$. The equation of the curve is $y = Cx^2$.

7. Orthogonal trajectories using symbol manipulation software.

Neither Maple nor Mathematica can obtain symbolic solutions to the differential equation that defines the orthogonal trajectories in these problems. The following example–problem (a)–shows how to display numerically generated trajectories using Maple. The code for Mathematica is quite similar.

1.7 Homogeneous Equations

The letters M and N will always refer to the equation $Mdx + Ndy = 0$. Note that this equation is equivalent to $y' = -\frac{M}{N}$.

1. Verify the equation is homogeneous, solve.

(a) $M = x^2 - 2y^2$ and $N = xy$ are homogeneous of degree 2; $y' = -\frac{x^2-2y^2}{xy}$. The substitutions $z = \frac{y}{x}$ and $y' = xz' + z$ yield the separable equation $xz' + z = 2z - \frac{1}{z}$. This equation has solutions $z = \pm\sqrt{1 + Cx^2}$ so the original equation has solutions $y = \pm x\sqrt{1 + Cx^2}$.

(b) $M = 3xy + 2y^2$ and $N = -x$. M has degree 2 and N has degree 1. This equation is not homogeneous. Note that it is Bernoulli.

(c) Divide the equation by x^2 to obtain $y' = 3(1 + (\frac{y}{x})^2 \arctan\frac{y}{x} + \frac{y}{x}$, a homogeneous equation. The substitutions $z = \frac{y}{x}$ and $y' = xz' + z$ yield the separable equation $xz' + z = 3(1 + z^2)\arctan z + z$. This equation has the solution $z = \tan(Cx^3)$ so the original equation has solution $y = x\tan(Cx^3)$.

(d) Divide the equation by x to make it homogeneous. The substitutions $z = \frac{y}{x}$ and $y' = xz' + z$ yield the separable equation $(xz' + z)\frac{\sin z}{z} = \sin z + \frac{1}{z}$. This equation has the solution $\cos z + \ln cx = 0$ so the original equation has solution $\cos\frac{y}{x} + \ln cx = 0$.

(e) Divide by x to make the equation homogeneous. The substitutions $z = \frac{y}{x}$ and $y' = xz' + z$ yield the separable equation $xz' + z = z + 2e^{-z}$. This equation has the solution $z = \ln(2\ln x + C)$ so the original equation has solution $y = x\ln(2\ln x + C)$.

(f) $M = x - y$ and $N = -(x + y)$ are homogeneous of degree 1; $y' = \frac{x-y}{x+y}$. The substitutions $z = \frac{y}{x}$ and $y' = xz' + z$ yield the separable equation $xz' + z = \frac{1-z}{1+z}$. This equation has the solution $z^2 + 2z - 1 = Cx^{-2}$ and the original equation has solution $y^2 + 2xy - x^2 = C$.

(g) This equation is linear and homogeneous. To solve it as a homogeneous equation, divide by x: $y' = 2 - \frac{6y}{x}$, then substitute $z = \frac{y}{x}$ to obtain the separable equation $xz' + z = 2 - 6z$. The solution is $z = \frac{2}{7} + Cx^{-7}$ so the original equation has the solution $y = \frac{2x}{7} + Cx^{-6}$.

(h) Divide by x: $y' = \sqrt{1 + \frac{y^2}{x^2}}$, then substitute $z = \frac{y}{x}$ to obtain the separable equation $xz' + z = \sqrt{1+z^2}$. The solution is $z^2 + z\sqrt{1+z^2} + \ln(z + \sqrt{1+z^2}) = 2\ln x + C$. The solution to the original equation can be simplified to $y^2 + y\sqrt{x^2+y^2} + x^2\ln(y + \sqrt{x^2+y^2}) = x^2(3\ln x + C)$.

(i) The equation is Bernoulli and homogeneous. To solve it as homogeneous, divide by x^2: $y' = \frac{y^2}{x^2} + \frac{2y}{x}$, and substitute $z = \frac{y}{x}$ to obtain the separable equation $xz' + z = z^2 + 2z$. The solution is $z = \frac{x}{C-x}$. The solution to the original equation is $y = \frac{x^2}{C-x}$.

(j) $M = x^3 + y^3$ and $N = -xy^2$ are homogeneous of degree 3; $y' = \frac{x^3+y^3}{xy^2}$. The substitution $z = \frac{y}{x}$ yields the separable equation $xz' + z = \frac{1+z^3}{z^2}$. This equation has the solution $z = \sqrt[3]{3\ln x + C}$ and the original equation has the solution $y = x\sqrt[3]{3\ln x + C}$.

3. Solving $\frac{dy}{dx} = F(\frac{ax+by+c}{dx+ey+f})$

(a) The substitutions $x = z - h$ and $y = w - k$ yield the equation $\frac{dw}{dz} = F(\frac{az+bw-(ah+bk-c)}{dz+ew-(dh+ek-f)})$. Note that $\frac{dy}{dx} = \frac{dy}{dw}\cdot\frac{dw}{dz}\cdot\frac{dz}{dx} = \frac{dw}{dz}$. Since $ae \neq bd$ there are unique numbers h and k such that $ah + bk = c$ and $dh + ek = f$. Using these values the new equation is $\frac{dw}{dz} = F(\frac{az+bw}{dz+ew})$ which is homogeneous.

(b) If $ae = bd$, then there is a constant k such that $cx + dy = k(ax + by)$ for all x and y. We are assuming that a and b are not both 0. The equation then has the form $\frac{dy}{dx} = F(\frac{ax+by+c}{k(ax+by)+f})$. The substitution $z = ax + by$ reduces it to $z' = a + bF(\frac{z+c}{kz+f})$ which is separable.

5. Observe that if $z = \frac{y}{x^n}$ then $y' = x^n z' + nx^{n-1}z$.

(a) The substitution $z = \frac{y}{x^n}$ yields $z' = \frac{x^{-(2n+1)} - z^2(2n+1)}{2xz}$. Setting $n = -\frac{1}{2}$ this becomes $z' = \frac{1}{2xz}$, a separable equation with solution $z^2 = \ln x + C$. Therefore, $y^2 = \frac{\ln x + C}{x}$.

(b) The substitution $z = \frac{y}{x^n}$ yields $z' = \frac{2x^{-(2n+1)}+z^2(3-4n)}{4xz}$. Setting $n = \frac{3}{4}$ this becomes $z' = \frac{2x^{-5/2}}{4xz}$, a separable equation with solution $z^2 = -\frac{2}{5}x^{-5/2} + C$. Therefore, $y^2 = Cx^{3/2} - 2x^{-1}/5$. (The solution can be also be obtained using $n = -1/2$.)

(c) The substitution $z = \frac{y}{x^n}$ yields $z' = -z\frac{(n-1)x+x^{n+1}z(n+1)}{x(x+x^{n+2}z)}$. Setting $n = -1$ this becomes $z' = \frac{2z}{x(1+z)}$, a separable equation with solution $z = \ln z = 2\ln x + C$. Therefore, $xy + \ln y = \ln x + C$.

7. Solving homogeneous equations using symbol manipulation software.

Neither Maple nor Mathematica can obtain complete symbolic solutions to these differential equations. The following example shows Maple's answer to problem (a). Mathematica's answer is quite similar.

1.8 Integrating Factors

The letters M and N will always refer to the equation $Mdx + Ndy = 0$.

1. Solve by finding an integrating factor.

(a) $M = -2xy$ and $N = 3x^2 - y^2$; $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -\frac{4}{y}$. An integrating factor is $\mu(y) = \frac{1}{y^4}$, and the solution is $x^2 - y^2 = Cy^3$.

(b) $M = xy - 1$ and $N = x^2 - xy$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x}$. An integrating factor is $\mu(x) = \frac{1}{x}$, and the solution is $2xy - 2\ln x - y^2 = C$.

(c) $M = y$ and $N = -x - 3x^3y^4$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{2+9x^3y^4}{x(1+3x^3y^4)}$, and $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -\frac{2+9x^3y^4}{y}$. Consequently, there are no integrating factors that are function of x alone or functions of y alone. Try a factor of the form $\mu = \frac{1}{(xy)^n}$. This is motivated by the observation that the equation becomes $\frac{xdy+ydx}{(xy)^n} = -\frac{3x^3y^4dy}{(xy)^n}$ or $\frac{d(xy)}{(xy)^n} = -\frac{3x^3y^4dy}{(xy)^n}$. The left side is directly integrable for any n and, when $n = 3$, so is the right side: $\frac{d(xy)}{(xy)^3} = -3ydy$. The solution is $\frac{-1/2}{(xy)^2} = -\frac{3}{2}y^2 + C$ or $(xy)^{-2} = 3y^2 + C$.

(d) $M = e^x$ and $N = e^x \cot y + 2y\csc y$; $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \cot y$. An integrating factor is $\mu(y) = \sin y$, and the solution is $y^2 + e^x \sin y = C$.

(e) $M = (x+2)\sin y$ and $N = x\cos y$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 1 + \frac{1}{x}$. An integrating factor is $\mu(x) = xe^x$, and the solution is $x^2 e^x \sin y = C$.

(f) This equation is similar to the one in part (c). Write it in the form $xdy + ydx = 2x^2y^3dy$ and divide by $(xy)^2$ to get $\frac{d(xy)}{(xy)^2} = 2ydy$. The solution is $\frac{1}{xy} = -y^2 + C$.

(g) $M = x + 3y^2$ and $N = 2xy$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2}{x}$. An integrating factor is $\mu(x) = x^2$, and the solution is $x^4 + 4x^3y^2 = C$.

(h) $M = y$ and $N = 2x - ye^y$; $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{1}{y}$. An integrating factor is $\mu(y) = y$ and the solution is $xy^2 - e^y(y^2 - 2y + 2) = C$.

(i) $M = y\ln y - 2xy$ and $N = x + y$; $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -\frac{1}{y}$. An integrating factor is $\mu(y) = \frac{1}{y}$ and the solution is $y - x^2 + x\ln y = C$.

(j) $M = y^2 + xy + 1$ and $N = x^2 + xy + 1$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y-x}{x^2+xy+1}$, and $-\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{x-y}{y^2+xy+1}$. Consequently, there are no integrating factors that are functions of x alone or functions of y alone. Let's see if we can take advantage of the symmetry of the equation. Start by expressing it in the form $dx + dy + y(y+x)dx + x(x+y)dy = 0$, then divide by $x+y$: $\frac{dx+dy}{x+y} + ydx + xdy = 0$. This is equivalent to $\frac{d(x+y)}{x+y} + d(xy) = 0$ and the solution, obtained by integrating is $\ln(x+y) + xy = C$. Note. An integrating factor for this equation is $\mu = (x+y)^{-1}$. The reader is invited to check that the equation $\frac{dx+dy}{x+y} + ydx + xdy = 0$ is exact and then solve it by the method of exact equations.

(k) $M = x^3 + xy^3$ and $N = 3y^2$; $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = x$. An integrating factor is $\mu(x) = e^{x^2/2}$ and the solution is $e^{x^2/2}(y^3 + x^2 - 2) = C$.

3. If $\mu(Mdx + Ndy) = 0$ is exact, then $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$. Therefore, $\mu\frac{\partial M}{\partial y} + \frac{\partial \mu}{\partial y}M = \mu\frac{\partial N}{\partial x} + \frac{\partial \mu}{\partial x}N$. This can be written as $\mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = \frac{\partial \mu}{\partial x}N - \frac{\partial \mu}{\partial y}M$. If $\mu = \mu(x+y)$, then $\frac{\partial \mu}{\partial x} = \frac{\partial \mu}{\partial y} = \mu'(x+y)$ so the last equation can be rearranged to

$$\frac{\mu'(x+y)}{\mu(x+y)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N - M}.$$

Therefore, there is an integrating factor of the form $\mu(x+y)$ if and only if $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N-M}$ is a function of the form $f(x+y)$. If this is the case, the integrating factor is $\mu(z)=e^{\int f(z)dz}$ where $z=x+y$. See Exercise 1 (j) above.

1.9 Reduction of Order

1. Solve using reduction of order.

(a) Make the substitution $y'=p$, $y''=p\frac{dp}{dy}$ to obtain $yp\frac{dp}{dy}=-p^2$. Separate the variables: $\frac{dp}{p}=-\frac{dy}{y}$, and integrate to get $\ln p=-\ln y+C$. Equivalently, $p=C/y$. To finish replace p with $y'(=\frac{dy}{dx})$ and integrate once more: $\frac{dy}{dx}=\frac{C}{y}$, $ydy=Cdx$, and $\frac{y^2}{2}=Cx+D$. This can be expressed simply as $y^2=Cx+D$.

(b) Make the substitution $y'=p$ to obtain $xp'=p+p^3$, a separable equation. Separate to $\frac{dp}{p^3+p}=\frac{dx}{x}$ and integrate (partial fractions). The solution is $p=\pm\frac{x}{\sqrt{C-x^2}}$. Replace p with y' and integrate once more for the solutions: $y=\pm\sqrt{C-x^2}+D$.

(c) Proceed as in Example 1.10.2 to obtain $y=A\sinh(kx)+B\cosh(kx)$.

(d) Make the substitution $y'=p$ to obtain $x^2p'=2xp+p^2$, a Bernoulli equation ($n=2$) with solution $p=\frac{x^2}{C-x}$. Replace p with y' and integrate for the solution: $y=-\frac{x^2}{2}-Cx-C^2\ln(x-C)+D$.

(e) Make the substitution $y'=p$, $y''=p\frac{dp}{dy}$ to obtain $2yp\frac{dp}{dy}=1+p^2$, a separable equation with solution $p=\pm\sqrt{Cy-1}$. Replace p with $\frac{dy}{dx}$ to obtain another separable equation that integrates to $Cx=\pm 2\sqrt{Cy-1}+D$.

(f) Make the substitution $y'=p$, $y''=p\frac{dp}{dy}$ to obtain $yp\frac{dp}{dy}=p^2$, a separable equation with solution $p=Cy$. Replace p with $\frac{dy}{dx}$ to obtain another separable equation that integrates to $y=De^{Cx}$.

(g) Make the substitution $y'=p$ to obtain $xp'+p=4x$, a linear equation with solution $p=2x+Cx^{-1}$. Replace p with $\frac{dy}{dx}$ and integrate once more for the solution: $y=x^2+C\ln x+D$.

3. Solve using both methods: x missing, y missing. Reconcile the results.

(a) $y'' = 1 + (y')^2$

x missing Substitute $y' = p$, $y'' = p'$ to get $p' = 1 + p^2$ (separable). The solution is $p = \tan(x + C)$; that is, $y' = \tan(x + C)$. Integrate to obtain $y = \ln(\sec(x + C)) + D$.

y missing Substitute $y' = p$, $y'' = p\frac{dp}{dy}$ to get $p\frac{dp}{dy} = 1 + p^2$ (separable). The solution is $p = \sqrt{Ce^{2y} - 1}$; that is, $y' = \sqrt{Ce^{2y} - 1}$. Integrate to obtain $\arctan\sqrt{Ce^{2y} - 1} = x + D$. This is equivalent to $\sqrt{Ce^{2y} - 1} = \tan(x + D)$.

To reconcile these solutions note that the second one is equivalent to $e^{2y} = C\sec^2(x + D)$.

(b) $y'' + (y')^2 = 1$

x missing Substitute $y' = p$, $y'' = p'$ to get $p' = 1 - p^2$ (separable). The solution is $p = \tanh(x + C)$; that is, $y' = \tanh(x + C)$. Integrate to obtain $y = \ln(\cosh(x + C)) + D$.

y missing Substitute $y' = p$, $y'' = p\frac{dp}{dy}$ to get $p\frac{dp}{dy} = 1 - p^2$ (separable). The solution is $p = \sqrt{Ce^{-2y} + 1}$; that is, $y' = \sqrt{Ce^{-2y} + 1}$. Integrate to obtain $\text{arctanh}\sqrt{Ce^{-2y} + 1} = x + D$. This is equivalent to $\sqrt{Ce^{-2y} + 1} = \tanh(x + D)$.

To reconcile these solutions note that the second one is equivalent to $e^{2y} = C\cosh^2(x + D)$.

1.10 The Hanging Chain and Pursuit Curves

1. The statement of the problem is incomplete. The relation $T = wy$ is valid provided the x-axis is placed so that $h_0 = \frac{T_0}{w}$, where T_0 denotes the tension at $x = 0$. This implies that

$$y = \frac{T_0}{w}\cosh(\frac{w}{T_0}x)$$

and $y' = \sinh(\frac{w}{T_0}x)$. Using the identity $\cosh(\frac{w}{T_0}x) = \sqrt{1 + \sinh^2(\frac{w}{T_0}x)}$ we have $\cosh(\frac{w}{T_0}x) = \sqrt{1 + (y')^2}$ which allows us to rearrange the displayed equation to $wy = T_0\sqrt(1+(y')^2)$. But $\cos\theta = \frac{1}{1+(y')^2}$ (see Figure 1.11) so the last equation implies that $wy = \frac{T_0}{\cos\theta} = T$.

3. The assumption is that the weight supported from the bottom of the cable to the point (x, y) equals $L_0 x$. This means that the displayed equation proceeding Equation (1.33) in this section can be changed to $y' = \sqrt{L_0} T_0 x$ where T_0 denotes the tension at the low point, $x = 0$. Consequently, $y = \sqrt{L_0} 2T_0 x^2 + h_0$. The cable hangs in the shape of a parabola.

5. Let δ denote the planar weight density of the curtain (weight per unit area). Using this, the weight supported by the cord from the low point, $x = 0$, to the point (x, y) is equal to $\delta \int_0^2 y(t)dt$. Arguing as in this section we are led to $y' = \frac{\delta}{T_0} \int_0^x y(t)dt$ where T_0 is the tension in the cord at $x = 0$. Differentiate to obtain $y'' = \frac{\delta}{T_0} y$ or $y'' - a^2 y = 0$ where $a = \sqrt{\delta / T_0}$. This second order equation can be solved by reduction of order as in Example 1.10.3 (y' is missing) to obtain $y = c_1 e^{ax} + c_2 e^{-ax}$. THe fact that $y'(0) = 0$ implies that $c_1 = c_2$ so the solution has the form $y = c(e^{ax} + e^{-ax})$. The value of the constant c is determined by the height, h_0, of the curtain at $x = 0$: $y = \frac{h_0}{2}(e^{ax} + e^{-ax})$. The shape is similar to that of a catenary (it is not a catenary, why not?)

7. The trajectories orthogonal to the pursuit curve have the differential equation $y' = \frac{x}{\sqrt{a^2 - x^2}}$. Solve to show that the family is defined by $x^2 + (y - c)^2 = a^2$.

1.11 Electrical Circuits

1. Since $I = \frac{E_0}{R}(1 - e^{-Rt/L})$ the theoretical maximum value for I is $\frac{E_0}{R}$. Half this value is attained when $1 - e^{-Rt/L} = 1/2$. Solve to obtain $t = (L \ln 2)/R$ seconds.

3. The current is controlled by $L\frac{dI}{dt} + RI = E$.

(a) The current is maximum or minimum when $\frac{dI}{dt} = 0$, implying that $RI = E$.

(b) Differentiate the controlling equation to obtain $E' = LI'' + RI'$. Since $I' = 0$ when I is a maximum or a minimum, $E' = LI''$ at such times. If the current is a minimum, then $I'' > 0$ so $E' > 0$ and E is increasing. If current is a maximum, then $I'' < 0$ so $E' < 0$ and E is decreasing.

5. The controlling equations are $L\frac{dI}{dt} + \frac{1}{C}Q = 0$, $Q(0) = Q_0$, and $I(0) = 0$. In terms of Q this is $LQ'' + \frac{1}{C}Q = 0$, $Q(0) = Q_0$, $Q'(0) = 0$. Write this as $Q'' + \frac{1}{LC}Q = 0$. Since Q' is missing reduction of order can be used to solve for Q (see Example **): $Q = A\sin\omega t + B\cos\omega t$ where $\omega = 1/\sqrt{LC}$. Differentiate to find I: $I = Q' = A\omega\cos\omega t - B\omega\sin\omega t$. The condition that $I(0) = 0$ forces $A = 0$. The condition $Q(0) = Q_0$ tells us that $B = Q_0$. Therefore, $Q = Q_0\cos(t/\sqrt{LC})$ and $I = -\frac{Q_0}{\sqrt{LC}}\sin(t/\sqrt{LC})$.

Chapter 2

Second-Order Linear Equations

2.1 Second-Order Linear Equations with Constant Coefficients

1. Find the general solution of each of the following differential equations. See the table.

3. The associated polynomial $r^2 + Pr + Q$ has roots $r = \frac{-P \pm \sqrt{P^2 - 4Q}}{2}$. Suppose that P and Q are both positive. Then $P^2 - 4Q \geq 0$ implies that the roots are real and negative so $y \to 0$ as $x \to \infty$ because both exponential terms in the solution have negative exponents. If $P^2 - 4Q < 0$, then the roots are complex with negative real part. Consequently, the solutions are of the form $y = Ae^{-Px/2} \cos \omega x + Be^{-Px/2} \sin \omega x$ and will oscillate towards 0 as $x \to \infty$. The other cases are handled similarly.

5. Euler's equidimensional equation Changing the independent variable using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\dot{y}$ where the dot indicates differentiation with respect to the new independent variable, z. Similarly, $y'' = \frac{d}{dx}(\frac{1}{x}\dot{y}) = \frac{1}{x} \cdot \frac{1}{x}\ddot{y} - \frac{1}{x^2}\dot{y} = \frac{1}{x^2}(\ddot{y} - \dot{y})$. Making these substitutions into $x^2y'' + pxy' + qy = 0$ yields $x^2 \cdot \frac{1}{x^2}(\ddot{y} - \dot{y}) + px \cdot \frac{1}{x}\dot{y} + qy = 0$ which simplifies to $\ddot{y} + (p-1)\dot{y} + qy = 0$, an equation with constant coefficients. If $y = \phi(z)$ is the general solution to this equation, then $y = \phi(\ln x)$ will be the general solution to the Euler equidimensional equation. Note that the solution is only valid for $x > 0$.

Table 2.1: The general solutions for Exercise 1.

	Assoc Poly	Roots	General Solution
(a)	r^2+r-6	$2,-3$	$Ae^{2x}+Be^{-3x}$
(b)	r^2+2r+1	$-1,-1$	$Ae^{-x}+Bxe^{-x}$
(c)	r^2+8	$\pm 2\sqrt{2}i$	$A\cos(2\sqrt{2}x)+B\sin(2\sqrt{2}x)$
(d)	$2r^2-4r+8$	$1\pm\sqrt{3}i$	$Ae^x\cos(\sqrt{3}x)+Be^x\sin(\sqrt{3}x)$
(e)	r^2-4r+4	$2,2$	$Ae^{2x}+Bxe^{2x}$
(f)	$r^2-9r+20$	$4,5$	$Ae^{4x}+Be^{5x}$
(g)	$2r^2+2r+3$	$-\frac{1}{2}\pm\frac{\sqrt{5}}{2}i$	$Ae^{-x/2}\cos(\sqrt{5}x/2)+Be^{-x/2}\sin(\sqrt{5}x/2)$
(h)	$4r^2-12r+9$	$3/2,3/2$	$Ae^{3x/2}+Bxe^{3x/2}$
(i)	r^2+r	$-1,0$	$Ae^{-x}+B$
(j)	$r^2-6r+25$	$3\pm 4i$	$e^{3x}(A\cos 4x+B\sin 4x)$
(k)	$4r^2+20r+25$	$\pm 5/2$	$Ae^{5x/2}+Be^{-5x/2}$
(l)	r^2+2r+3	$-1\pm\sqrt{2}i$	$Ae^{-x}\cos(\sqrt{2}x)+Be^{-x}\sin(\sqrt{2}x)$
(m)	r^2-4	± 2	$Ae^{2x}+Be^{-2x}$
(n)	$4r^2-8r+7$	$1\pm\frac{\sqrt{3}}{2}i$	$Ae^x\cos(\sqrt{3}x/2)+Be^x\sin(\sqrt{3}x/2)$
(o)	$2r^2+r-1$	$-1,1/2$	$Ae^{-x}+Be^{x/2}$
(p)	$16r^2-8r+1$	$1/4$	$Ae^{x/4}+Bxe^{x/4}$
(q)	r^2+4r+5	$-2\pm i$	$Ae^{-2x}\cos x+Be^{-2x}\sin x$
(r)	r^2+4r-5	$-5,1$	$Ae^{-5x}+Be^{x}$

(a) The z equation is $\ddot{y}+2\dot{y}+10y=0$ with solution $y=Ae^{-z}\cos 3z+Be^{-x}\sin(3z)$. The x equation has the solution $y=Ax^{-1}\cos(3\ln x)+Bx^{-1}\sin(3\ln x)$.

(b) First divide the equation by 2. The z equation is then $\ddot{y}+4\dot{y}+4y=0$ with solution $y=Ae^{-2z}+Bze^{-2z}$. The x equation has the solution $y=Ax^{-2}+Bx^{-2}\ln x$.

(c) The z equation is $\ddot{y}+\dot{y}-12y=0$ with solution $y=Ae^{-4z}+Be^{3z}$. The x equation has the solution $y=Ax^{-4}+Bx^{3}$.

(d) Divide the equation by 4. The z equation is then $\ddot{y}-\dot{y}-\frac{3}{4}y=0$ with solution $y=Ae^{-z/2}+Be^{3z/2}$. The x equation has the solution $y=Ax^{-1/2}+Bx^{3/2}$.

(e) The z equation is $\ddot{y}-4\dot{y}+4y=0$ with solution $Ae^{2z}+Bze^{2z}$. The x equation has the solution $y=Ax^2+Bx^2\ln x$.

(f) The z equation is $\ddot{y}+\dot{y}-6y=0$ with solution $y=Ae^{-3z}+Be^{2z}$. The x equation has the solution $y=Ax^{-3}+Bx^{2}$.

(g) The z equation is $\ddot{y}+\dot{y}+3y=0$ with solution $y=Ae^{-z/2}\cos\frac{\sqrt{11}z}{2}+Be^{-z/2}\sin\frac{\sqrt{11}z}{2}$. The x equation has the solution

$$y=Ax^{-\frac{1}{2}}\cos\frac{\sqrt{11}\ln x}{2}+Bx^{-\frac{1}{2}}\sin\frac{\sqrt{11}\ln x}{2}.$$

(h) The z equation is $\ddot{y}-2y=0$ with solution $y=Ae^{-\sqrt{2}z}+Be^{\sqrt{2}z}$. The x equation has the solution $y=Ae^{-\sqrt{2}}+Bx^{\sqrt{2}}$.

(i) The z equation is $\ddot{y}-16y=0$ with solution $y=Ae^{-4z}+Be^{4z}$. The x equation as the solution $y=Ax^{-4}+Bx^{4}$.

2.2 The Method of Undetermined Coefficients

1. Find the general solution of each of the following equations.

(a) The auxiliary roots are -5 and 2 so the homogeneous equation has the solution $y=Ae^{-5x}+Be^{2x}$. Try $y=\alpha e^{4x}$ as a particular solution. Substitute and simplify to obtain $18\alpha e^{4x}=6e^{4x}$ which implies that $\alpha=1/3$. The general solution is $y=Ae^{-5x}+Be^{2x}+\frac{1}{3}e^{4x}$.

(b) The auxiliary roots are $\pm 2i$ so the homogeneous equation has the solution $y = A\cos 2x + B\sin 2x$. Try $y = \alpha\cos x + \beta\sin x$ as a particular solution. Substitute and simplify to obtain $3\alpha\cos x + 3\beta\sin x = 3\sin x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos 2x + B\sin 2x + \sin x$.

(c) The auxiliary roots are $-5, -5$ so the homogeneous equation has the solution $y = Ae^{-5x} + Bxe^{-5x}$. Neither $y = \alpha e^{-5x}$ nor $y = \alpha xe^{-5x}$ can be a particular solution because they are solutions to the homogeneous equation. Try $y = \alpha x^2 e^{-5x}$ instead. Substitute and simplify to obtain $2\alpha e^{-5x} = 14e^{-5x}$ which implies that $\alpha = 7$. The general solution is $y = Ae^{-5x} + Bxe^{-5x} + 7x^2e^{-5x}$.

(d) The auxiliary roots are $1 \pm 2i$ so the homogeneous equation has the solution $y = Ae^x\cos 2x + Be^x\sin x$. Try $y = \alpha x^2 + \beta x + \gamma$ as a particular solution. Substitute and simplify to obtain $5\alpha x^2 + (5\beta - 4\alpha)x + 2\alpha - 2\beta + 5\gamma = 25x^2 + 12$ which implies that $\alpha = 5, \beta = 4, \gamma = 2$. The general solution is $y = Ae^x\cos 2x + Be^x\sin x + 5x^2 + 4x + 2$.

(e) The auxiliary roots are $-2, 3$ so the homogeneous equation has the solution $y = Ae^{-2x} + Be^{3x}$. The function $y = \alpha e^{-2x}$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha xe^{-2x}$ instead. Substitute and simplify to obtain $-5\alpha xe^{-2x} = 20e^{-2x}$ which implies that $\alpha = -4$. The general solution is $y = Ae^{-2x} + Be^{3x} - 4xe^{-2x}$.

(f) The auxiliary roots are $1, 2$ so the homogeneous equation has the solution $y = Ae^x + Be^{2x}$. Try $y = \alpha\cos 2x + \beta\sin 2x$ as a particular solution. Substitute and simplify to obtain $(-6\alpha - 2\beta)\cos 2x + (-2\alpha + 6\beta)\sin 2x = 14\sin 2x - 18\cos 2x$ which implies that $\alpha = 3$ and $\beta = 2$. The general solution is $y = Ae^x + Be^{2x} + 3\cos 2x + 2\sin 2x$.

(g) The auxiliary roots are $\pm i$ so the homogeneous equation has the solution $y = A\cos x + B\sin x$. The function $y = \alpha\cos x + \beta\sin x$ can not be a particular solution because it is a solution to the homogeneous equation. Try $y = \alpha x\cos x + \beta x\sin x$ instead. Substitute and simplify to obtain $2\beta\cos x - 2\alpha\sin x = 2\cos x$ which implies that $\alpha = 0$ and $\beta = 1$. The general solution is $y = A\cos x + B\sin x + x\sin x$.

(h) The auxiliary roots are $0, 2$ so the homogeneous equation has the solution $y = A + Be^{2x}$. The function $y = \alpha x + \beta$ is not a particular solution because part of it is a solution to the homogeneous equation. Try $y = \alpha x^2 + \beta x$ instead. Substitute and simplify to obtain $-4\alpha x + 2\alpha - 2\beta = 12x - 10$ which implies that $\alpha = -3$ and $\beta = 2$. The general solution is $y = A + Be^{2x} - 3x^2 + 2x$.

(i) The auxiliary roots are $1, 1$ so the homogeneous equation has the solution $y = Ae^x + Bxe^x$. Neither $y = \alpha e^x$ not $y = \alpha xe^x$ is a particular solution because they are both solutions to the homogeneous equation. Try $y = \alpha x^2 e^x$ instead. Substitute and simplify to obtain $2\alpha e^x = 6e^x$ which implies that $\alpha = 3$. The general solution is $y = Ae^x + Bxe^x + 3x^2 e^x$.

(j) The auxiliary roots are $1 \pm i$ so the homogeneous equation has the solution $y = Ae^x \cos x + Be^x \sin x$. This means that $y = \alpha e^x \cos x + \beta e^x \sin x$ can not be a particular solution. Try $y = \alpha xe^x \cos x + \beta xe^x \sin x$ instead. Substitute and simplify to obtain $2\beta e^x \cos x - 2\alpha e^x \sin x = e^x \sin x$ which implies that $\alpha = -1/2$ and $\beta = 0$. The general solution is $y = Ae^x \cos x + Be^x \sin x - \frac{1}{2} xe^x \cos x$.

(k) The auxiliary roots are $-1, 0$ so the homogeneous equation has the solution $y = Ae^{-x} + B$. This means that $y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon$ can not be a particular solution. Try $y = \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x$ instead. Substitute and simplify to obtain $5\alpha x^4 + (20\alpha + 4\beta)x^3 + (12\beta + 3\gamma)x^2 + (6\gamma + 2\delta)x + 2\delta + \epsilon = 10x^4 + 2$ which implies that $\alpha = 2, \beta = -10, \gamma = 40, \delta = -120$ and $\epsilon = 242$. The general solution is $y = Ae^{-x} + B + 2x^5 - 10x^4 + 40x^3 - 120x^3 + 242x$.

3. The auxiliary roots are $-k, k$ so the homogeneous equation has the solution $y = Ae^{kx} + Be^{-kx}$. Try $y = \alpha \cos bx + \beta \sin bx$ as a particular solution. Substitute and simplify to obtain $(k^2 - b^2)\alpha \cos bx + (k^2 - b^2)\beta \sin bx = \sin bx$ which implies that $\alpha = 0$ and $\beta = \frac{1}{k^2 - b^2}$. The general solution is $y = Ae^{kx} + Be^{-kx} + \frac{1}{k^2 - b^2} \sin bx$.

2.3 The Method of Variation of Parameters

1. Find a particular solution.

(a) The homogeneous solution is $y = A\sin 2x + B\cos 2x$ so the particular solution has the form $y = v_1\sin 2x + v_2\cos 2x$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'\sin 2x + v_2'\cos 2x &= 0 \\ 2v_1'\cos 2x - 2v_2'\sin 2x &= \tan 2x \end{aligned}$$

Therefore, $v_1' = \frac{1}{2}\sin x$ and $v_2' = -\frac{1}{2}\sin 2x\tan 2x$. Integrate to get $v_1 = -\frac{1}{4}\cos 2x$ and $v_2 = \frac{1}{4}\sin 2x - \frac{1}{4}\ln(\sec 2x + \tan 2x)$. Therefore, $y = -\frac{1}{4}\cos 2x\ln(\sec 2x + \tan 2x)$.

(b) The homogeneous solution is $y = Ae^{-x} + Bxe^{-x}$ so the particular solution has the form $y = v_1e^{-x} + v_2xe^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{-x} + v_2'xe^{-x} &= 0 \\ -v_1'e^{-x} + v_2'(e^{-x} - xe^{-x}) &= e^{-x}\ln x \end{aligned}$$

Therefore, $v_1' = -x\ln x$ and $v_2' = \ln x$. Integrate to get $v_1 = -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2$ and $v_2 = x\ln x - x$. Therefore, $y = \frac{1}{4}x^2e^{-x}(2\ln x - 3)$.

(c) The homogeneous solution is $y = Ae^{3x} + Be^{-x}$ so the particular solution has the form $y = v_1e^{3x} + v_2e^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{3x} + v_2'e^{-x} &= 0 \\ 3v_1'e^{-x} - v_2'e^{-x} &= 64xe^{-x} \end{aligned}$$

Therefore, $v_1' = 16xe^{-4x}$ and $v_2' = -16x$. Integrate to get $v_1 = -(4x+1)e^{-4x}$ and $v_2 = -8x^2$. Therefore, $y = -e^{-x}(8x^2+4x+1)$. The last term can be dropped since $-e^{-x}$ is a solution to the homogeneous solution.

(d) The homogeneous solution is $y = Ae^{-x}\sin 2x + Be^{-x}\cos 2x$ so the particular solution has the form $y = v_1e^{-x}\sin 2x + v_2e^{-x}\cos 2x$ where v_1 and v_2 satisfy the system

$$\begin{aligned} &v_1'e^{-x}\sin 2x + v_2'e^{-x}\cos 2x = 0 \\ &v_1'(-e^{-x}\sin 2x + 2e^{-x}\cos 2x) + v_2'(e^{-x}\cos 2x + 2e^{-x}\sin 2x) \\ &\qquad = e^{-x}\sec 2x \end{aligned}$$

Therefore, $v_1' = \frac{1}{2}$ and $v_2' = -\frac{1}{2}\tan 2x$. Integrate to get $v_1 = \frac{1}{2}x$ and $v_2 = \frac{1}{4}\ln(\cos 2x)$. Therefore, $y = \frac{1}{2}xe^{-x}\sin 2x + \frac{1}{4}e^{-x}\cos 2x \ln(\cos 2x)$.

(e) The homogeneous solution is $y = Ae^{-x/2} + Be^{-x}$ so the particular solution has the form $y = v_1e^{-x/2} + v_2e^{-x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{-x/2} + v_2'e^{-x} &= 0 \\ -\frac{1}{2}v_1'e^{-x/2} - v_2'e^{-x} &= \frac{1}{2}e^{-3x} \end{aligned}$$

Therefore, $v_1' = e^{-5x/2}$ and $v_2' = -e^{-2x}$. Integrate to get $v_1 = -\frac{2}{5}e^{-5x/2}$ and $v_2 = \frac{1}{2}e^{-x}$. Therefore, $y = \frac{1}{10}e^{-3x}$.

(f) The homogeneous solution is $y = Ae^{x} + Be^{2x}$ so the particular solution has the form $y = v_1e^{x} + v_2e^{2x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{x} + v_2'e^{2x} &= 0 \\ v_1'e^{x} + 2v_2'e^{2x} &= (1+e^{-x})^{-1} \end{aligned}$$

Therefore, $v_1' = -\frac{e^{-x}}{1+e^{-x}}$ and $v_2' = \frac{e^{-2x}}{1+e^{-x}}$. Integrate to get $v_1 = \ln(1+e^{-x})$ and $v_2 = \ln(1+e^{-x}) - e^{-x}$. Consequently, the particular solution is $y = (e^{x} + e^{2x})\ln(1+e^{-x}) - e^{x}$.

3. By Inspection The auxiliary polynomial is $r^2 - 2r + 1$ with roots $1, 1$ so the homogeneous solution is $y = Ae^{x} + Bxe^{x}$. Therefore, there is a particular solution of the form $y = \alpha x + \beta$. Substitute to find that $y = 2x + 4$ is a particular solution.

By Variation of Parameters The particular solution has the form $y = v_1e^{x} + v_2xe^{x}$ where v_1 and v_2 satisfy the system

$$\begin{aligned} v_1'e^{x} + v_2'xe^{x} &= 0 \\ v_1'e^{x} + v_2'(xe^{x} + e^{x}) &= 2x \end{aligned}$$

Therefore, $v_1' = -2x^2e^{-x}$ and $v_2' = 2xe^{-x}$. Integrate to get $v_1 = (2x^2 + 4x + 4)e^{-x}$ and $v_2 = -(2x+2)e^{-x}$. Therefore, $y = 2x + 4$.

5. The solution y_h to the homogeneous equation is given below. Use it to apply the variation of parameters technique to obtain the particular solution, labeled y_p. The general solution is $y = y_h + y_p$.

Warning Write the equation in the form

$$y'' + P(x)y' + Q(x)y = R(x)$$

before applying the variation of parameters algorithm.

(a) $y_h = Ax + B(x^2 + 1)$; $y_p = x^4/6 - x^2/2$.

(b) $y_h = Ax^{-1} + Be^x$; $y_p = -\frac{1}{3}x^2 - x - 1$.

(c) $y_h = Ax + Be^x$; $y_p = x^2 + x + 1$.

(d) $y_h = A(1 + x) + Be^x$; $y_p = \frac{1}{2}e^{2x}(x - 1)$.

(e) $y_h = Ax^2 + Bx$; $y_p = -xe^{-x} - (x^2 + x)\int \frac{e^{-x}}{x}dx$.
To obtain this solution formula you will have to apply integration by parts to $\int \frac{e^{-x}}{x^2}dx$. The remaining integral does not evaluate to an elementary function.

2.4 The Use of a Known Solution to Find Another

1. Find y_2 and the general solution, given y_1.

(a) Since $p(x) = 0$, $e^{-\int p(x)dx} = e^0 = 1$ and $y_2(x) = \sin(x)v(x)$ where $v(x) = \int \frac{1}{\sin^2 x}dx = \int \csc^2 x dx = -\cot x$. Therefore, $y_2(x) = -\cos x$. The general solution is $y = A\sin x + B\cos x$.

(b) Once more, $p(x) = 0$ and $e^{-\int p(x)dx} = e^0 = 1$. Therefore, $y_2(x) = e^x v(x)$ where $v(x) = \int \frac{1}{e^{2x}}dx = -\frac{1}{2}e^{-2x}$. Therefore, $y_2(x) = e^x(-\frac{1}{2}e^{-2x}) = -\frac{1}{2}e^{-x}$. The general solution is $y = Ae^x + Be^{-x}$.

3. If $y = y_1 = x^2$, then $x^2y'' + xy' - 4y = x^2 \cdot 2 + x \cdot 2x - 4 \cdot x^2 = 0$. To find y_2 observe that $p(x) = 1/x$ and $e^{\int 1/x dx} = e^{-\ln x} = 1/x$. Therefore, $y_2 = x^2 \int \frac{1}{x^4} \cdot \frac{1}{4} = x^2 \cdot \frac{-4}{x^4} = -4/x^2$. The general solution is $y = Ax^2 + Bx^{-2}$.

5. If $y = y_1 = x^{-1/2}\sin x$, then

$$y' = x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x$$
$$y'' = -x^{-1/2}\sin x - x^{-3/2}\cos x + \frac{3}{4}x^{-5/2}\sin x$$

Substitute carefully into (*).

To find y_2 observe that $p(x) = 1/x$ so $e^{-\int 1/x dx} = 1/x$. Therefore, $y_2 = x^{-1/2}\sin x \int \frac{1}{x^{-1}\sin^2 x}\cdot\frac{1}{x}dx = x^{-1/2}\sin x \int \csc^2 x dx = x^{-1/2}\sin x(-\cot x) = -x^{-1/2}\cos x$. Therefore, the general solution is $y = Ax^{-1/2}\sin x + Bx^{-1/2}\cos x$.

7. By inspection, $y = y_1 = x$ is one solution. Since $p(x) = -xf(x)$, the second solution has the form $y_2 = x\int \frac{1}{x^2}e^{\int xf(x)dx}dx$. The general solution has the form $y = Ax + Bx\int \frac{1}{x^2}e^{\int xf(x)dx}dx$.

9. If y_1 and y_2 are linearly dependent, then the function $v(x)$ is a constant and has a derivative that is identically 0. However, $v'(x) = \frac{1}{y_1^2}e^{-\int P(x)dx}$, which is never 0 (exponentials cannot vanish).

2.5 Vibrations and Oscillations

1. The amplitude $A = \frac{F_0}{\sqrt{(k-\omega^2 M)^2+\omega^2 c^2}}$ attains its maximum at the ω value that minimizes the polynomial $\phi(\omega) = (k - \omega^2 M)^2 + \omega^2 c^2$. A simple calculation will show that $\phi'(\omega) = 0$ when $\omega = 0$ or $\omega = \pm\sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$. Thus if $\frac{k}{M} \leq \frac{c^2}{2M^2}$, i.e. $c \geq \sqrt{2kM}$, then there is no resonance frequency and as ω increases from 0, the amplitude A will steadily decrease to 0. On the other hand, if $0 < c < \sqrt{2kM}$, then A will increase as ω increases reaching its maximum value at the $\omega^* = \sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$ and decrease to 0 thereafter. The resonance frequency is $\frac{1}{2\pi}\sqrt{\frac{k}{M} - \frac{c^2}{2M^2}}$. This frequency is clearly less than the natural frequency $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$.

3. Let b denote the density of the buoy (weight per unit volume) and ω the density of water. The volume of the buoy is $V = \frac{4}{3}\pi r^3$.

> The volume of a slice of the buoy from its center to a point y units from center is $\pi r^2 y - \frac{1}{3}y^3$ (exercise).

Since the buoy floats half-submerged, $b = \omega/2$. As it bobs up and down let y be the distance from its center to the surface of the water (up is positive). If $y > 0$, then the net force on the buoy is negative given by the difference between the upward buoyant force of the water:

$$w \cdot (\frac{V}{2} - \pi r^2 y + \frac{1}{3}\pi y^3),$$

and the downward weight of the buoy $b \cdot V$. Subtracting, the net force is $\omega(-\pi r^2 y + \frac{1}{3}\pi y^3)$. Newton's law ($ma = F$) applied to the sphere, at its center of mass, yields the following equation (g is the gravitational constant)

$$\frac{b \cdot V}{g} y'' = -\pi \omega r^2 y + \frac{1}{3}\omega \pi y^3.$$

This is a second-order non-linear differential equation. However, if the buoy is only "slightly" depressed, then the linearized version (ignore the y^3 term) provides an excellent model for the motion. The linearized equation simplifies to $y' + a^2 y = 0$ where $a = \sqrt{\frac{3g}{2r}}$. The period of the motion is $2\pi\sqrt{\frac{2r}{3g}}$ seconds.

5. Recall, Section 1.10 problem 4, that inside the Earth the force of gravity on an object is proportional to its distance from the center. Let x be the distance from the train to the center of a tunnel of length $2L$. Draw a picture to see that the distance from the train to the center of the Earth is $\sqrt{x^2 + R^2 - L^2}$ where R is the radius of the Earth. The magnitude of the force on the train, in the direction of the center of the Earth, is then $F_c = k\sqrt{x^2 + R^2 - L^2}$. The value of k can be found from this equation when the train is at the surface of the Earth: $mg = kR$, so $k = mg/R$.

The magnitude of the force on the train parallel to the tracks is the component of F_c in that direction: $F_c \cdot \cos\theta = F_c \cdot \frac{x}{\sqrt{x^2+R^2-L^2}} = kx$. When x is positive, the force is negative. Applying Newton's Second Law we have $mx'' = -kx = -\frac{mg}{R}x$. Thus $x'' + \frac{g}{R}x = 0$, and the period of motion is independent of L: $T = 2\pi\sqrt{\frac{R}{g}}$ seconds; this is approximately 90 minutes. The equation of motion for a particular L value is found from the initial conditions: $x(0) = L$ and $x'(0) = 0$. This yields $x(t) = L\cos\sqrt{\frac{g}{R}}t$. The greatest speed is $|x'(T/4)| = \sqrt{\frac{g}{R}}L \approx 4.43L$ miles per hour.

2.6 Newton's Law of Gravitation and Kepler's Laws

1. Kepler's Third

(a) In astronomy the semi-major axis of the orbit is called the *mean distance* to the Sun because it is the average of the least and greatest values of r. Let a_u and T_u denote the semi-major axis and period of Uranus. These are known from Example 2.6.1. According to Kepler's Third Law, $\frac{T_u^2}{a_u^3} = \frac{T_m^2}{a_m^3}$ where a_m and T_m are Mercury's semi-major axis and period. Consequently, being careful with the units–see Example 2.6.1–we have

$$a_m = (\frac{T_m}{T_u})^{2/3} \cdot a_u = (\frac{\frac{88}{365} \cdot (3.16 \times 10^7)}{26.516 \times 10^8})^{2/3} \cdot (2.87 \times 10^{14})$$
$$= 5.800 \times 10^{12} \text{ centimeters.}$$

This is 5.800×10^{12} centimeters.

(b) When distance is measured in astronomical units and time in years, then $\frac{4\pi^2}{GM} = 1$ (verify). Therefore, in this system of units, $T^2 = a^3$. For example, the value of a_m calculated above can also be found (in astronomical units) using $a_m = T_m^{2/3} = (\frac{88}{365})^{2/3} = 0.3874$ au. Multiply by $93,000,000$ to obtain $a_m = 36,000,000$ miles.
Regarding Saturn, $T_s = a_s^{3/2} = (9.54)^{3/2} = 29.5$ years.

3. According to Exercise 2, in the instant after the explosion, the motion of every particle that moves into an elliptical orbit about the Sun obeys the equation $v^2 = GM(\frac{2}{r} - \frac{1}{a})$. Consequently all of these particles move in an orbit with the same semi-major axis, a astronomical units, and (according to Kepler's Third Law) the same period, $T = a^{3/2}$ years. This means that T years later all of them will return to their original positions.

5. See Exercise 1, part (b).

(a) $T = 2^{3/2} = 2.83$ years.

(b) $T = 3^{3/2} = 5.20$ years.

(c) $T = 25^{3/2} = 125$ years.

Table 2.2: Solutions for 1-15.

	Associated Polynomial	General Solution
1.	$r(r-1)(r-2)$	$y = A + Be^x + Ce^{2x}$
3.	$(r-1)(r^2+r+1)$	$y = Ae^x + e^{-x/2}(B\cos(\sqrt{3}x/2) + C\sin(\sqrt{3}x/2))$
5.	$(r+1)^3$	$y = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x}$
7.	$(r^2-1)(r^2+1)$	$y = Ae^x + Be^{-x} + C\cos x + D\sin x$
9.	$(r-a)^2(r+a)^2$	$y = Ae^{ax} + Bxe^{ax} + Ce^{-ax} + Dxe^{-ax}$
11.	$(r+1)^2(r^2+1)$	$y = Ae^{-x} + Bxe^{-x} + C\cos x + D\sin x$
13.	$(r-1)(r-2)(r-3)$	$y = Ae^x + Be^{2x} + Ce^{3x}$
15.	$(r-6)(r-2)^2(r+2)^2$	$y = Ae^{6x} + Be^{2x} + Cxe^{2x} + De^{-2x} + Exe^{-2x}$

2.7 Higher-Order Coupled Harmonic Oscillators

1-15. Find the general solution. See Table **??**.

17. The associated polynomial is $r(r-1)(r-2)$ so the general solution to the homogeneous equation is $y_g = A+Be^x+Ce^{2x}$. Based on the forcing function our fist choice for y_p is $y = A + Be^{3x}$. However, this will not work because $y = A$ is a solution to the homogeneous equation. Try $y = Ax + Be^{3x}$ instead. Substitute this into the forced equation to see that $A = 5$ and $B = 7$. The general solution is $y = A+Be^x+Ce^{2x}+5x+7e^{3x}$.

19. The Euler Equidimensional Equation (order 3) Using $x = e^z$ is equivalent to $z = \ln x$ so $y' = \frac{1}{x}\dot{y}$ and $y'' = \frac{1}{x^2}(\ddot{y}-\dot{y})$. The dot indicates differentiation with respect to the new independent variable, z. See Section 2.1 problem 5. For the third derivative,

$$y''' = \frac{d}{dx}(\frac{1}{x^2}(\ddot{y}-\dot{y})) = \frac{1}{x^2}(\dddot{y}-\dot{y})\frac{1}{x} - \frac{2}{x^3}(\ddot{y}-\dot{y})$$
$$\frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}).$$

Making these substitutions into $x^3y''' + a_2x^2y'' + a_1xy' + a_0y = 0$ yields $x^3 \cdot \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}) + a_2x^2 \cdot \frac{1}{x^2}(\ddot{y}-\dot{y}) + a_1x \cdot \frac{1}{x}\dot{y} + a_0y = 0$ which simplifies to

$$\dddot{y} + (a_2-3)\ddot{y} + (a_1-a_2+2)\dot{y} + a_0y = 0,$$

an equation with constant coefficients. If $y = \phi(z)$ is the general solution to the equation, then $y = \phi(\ln x)$ will be the general solution to the Euler equidimensional equation. Note that this solution is only valid for $x > 0$.

(a) The z equation is $\dddot{y} - \dot{y} = 0$ with associated polynomial $r^3 - r = r(r^2 - 1)$. The solution is $y = A + Be^x + Ce^{-x}$ so the solution to the original equation is $y = A + Bx + Cx^{-1}$.

(b) The z equation is $\dddot{y} - 2\ddot{y} - \dot{y} + 2y = 0$ with associated polynomial $r^3 - 2r^2 - r + 2 = (r-1)(r+1)(r-2)$. The solution is $y = Ae^z + Be^{-z} + Ce^{2z}$ so the solution to the original equation is $y = Ax + Bx^{-1} + Cx^2$.

(c) The z equation is $\dddot{y} - \ddot{y} + \dot{y} - y = 0$ with associated polynomial $r^3 - r^2 + r - 1 = (r-1)(r^2+1)$. The solution is $y = Ae^z + B\cos z + C\sin z$ so the solution to the original equation is $y = Ax + B\cos(\ln x) + C\sin(\ln x)$.

21. The equation is

$$m_1 m_2 \frac{d^4 x_1}{dt^4} + (m_1(k_2+k_3) + m_2(k_1+k_3))\frac{d^2 x_1}{dt^2} + (k_1k_2 + k_1k_3 + k_2k_3)x_1 = 0.$$

Chapter 3

Power Series Solutions and Special Functions

3.1 Introduction and Review of Power Series

1. For the series (1): $\sum_{j=1}^{\infty} j! \cdot x^j$, $\lim_{j\to\infty} \left|\frac{(j+1)!x^{j+1}}{j!x^j}\right| = \lim_{j\to\infty}(j+1)|x| = \infty$ when $x \neq 0$. The series converges only when $x = 0$, $R = 0$.

For the series (2): $\sum_{j=0}^{\infty} x^j/j!$, $\lim_{j\to\infty} \left|\frac{x^{j+1}/(j+1)!}{x^j/j!}\right| = \lim_{j\to\infty} \frac{|x|}{j+1} = 0$ for all x. The series converges for all x, $R = \infty$.

For the series (3): $\sum_{j=0}^{\infty} x^j$, $\lim_{j\to\infty} |\frac{x^{j+1}}{x^j}| = \lim_{j\to\infty} |x| = |x|$. The series converges when $|x| < 1$ and diverges when $|x| > 1$, $R = 1$.

3. $\sin x = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^{2j-1}}{(2j-1)!}$;

$$\lim_{j\to\infty} \left|\frac{x^{2(j+1)-1}/(2(j+1)-1)!}{x^{2j-1}/(2j-1)!}\right| = \lim_{j\to\infty} \frac{|x^2|}{(2j+1)2j} = 0.$$

$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}$;

$$\lim_{j\to\infty} \left|\frac{x^{2(j+1)}/(2(j+1))!}{x^{2j}/(2j)!}\right| = \lim_{j\to\infty} \frac{|x|^2}{(2j+2)(2j+1)} = 0.$$

5. If $|x| < 1$, then $\lim_{n\to\infty} x^{n+1} = 0$ so

$$\lim_{n\to\infty} \sum_{j=0}^{n} x^j = \lim_{n\to\infty} \left(\frac{1-x^{n+1}}{1-x}\right) = \frac{1}{1-x}.$$

Replace x with $-x$ to verify that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$.

Integrate $\frac{1}{1+x} = 1-x+x^2-x^3+\cdots$ to get $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$. Then replace x with x^2 and integrate to obtain $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots$.

7. (a) If $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, then $y' = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$ and

$$y'' = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots = -y.$$

(b) $\lim_{j\to\infty} \left| \frac{x^{2(j+1)}/(2^2 \cdot 4^2 \cdots ((2(j+1))^2)}{x^{2j}/(2^2 \cdot 4^2 \cdots (2j)^2)} \right| = \lim_{j\to\infty} \frac{|x^2|}{(2j+2)^2} = 0.$

Starting with $y = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^2 \cdot 4^2 \cdots (2j)^2}$ we have

$$xy = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2^2 \cdot 4^2 \cdots (2j)^2}$$
$$y' = \sum_{j=1}^{\infty} \frac{(-1)^j \cdot 2j \cdot x^{2j-1}}{2^2 \cdot 4^2 \cdots (2j)^2}$$
$$xy'' = \sum_{j=1}^{\infty} \frac{(-1)^j \cdot 2j \cdot (2j-1) \cdot x^{2j-1}}{2^2 \cdot 4^2 \cdots (2j)^2}$$

Observe that $xy'' + y' = \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j-1}}{2^2 \cdot 4^2 \cdots (2(j-1))^2}$. If the sum is reindexed by replacing j with $j+1$, then it its the negative of the sum for xy.

3.2 Series Solution of First-Order Differential Equations

1. Find a power series solution of the form $\sum_j a_j x^j$. Then solve directly using methods from earlier parts of the book. All of the details for the first solution are given. The calculations for equations (b)-(f) are given in less detail. In all cases, the first step is substitution of $y = \sum_{j=0}^{\infty} a_j x^j$.

(a) Since $y' = \sum_{j=1}^{\infty} ja_jx^{j-1}$ and $2xy = \sum_{j=0}^{\infty} 2a_jx^{j+1}$. Writing the differential equation in the form $y' - 2xy = 0$ we have

$$\sum_{j=1}^{\infty} ja_jx^{j-1} - \sum_{j=0}^{\infty} 2a_jx^{j+1} = 0.$$

Reindex the second sum, $j \to j-2$:

$$\sum_{j=1}^{\infty} ja_jx^{j-1} - \sum_{j=2}^{\infty} 2a_{j-2}x^{j-1} = 0,$$

then split off the first term from the first sum and move it to the right side

$$\sum_{j=2} ja_jx^{j-1} - \sum_{j=2}^{\infty} x^{j-1} = a_1.$$

Equivalently,

$$\sum_{j=2}^{\infty}(ja_j - 2a_{j-2})x^{j-1} = a_1.$$

It follows that $a_1 = 0$ and $ja_j - 2a_{j-2}) = 0$ for $j \geq 2$. This is the recursion relation; a_0 can be chosen arbitrarily.

Written in the form $a_j = \frac{2}{j}a_{j-2}$ the recursion relation implies that $a_j = 0$ if j is odd. For the even coefficients let $a_0 = A$. Then $a_2 = \frac{2}{2}A = A$, $a_4 = \frac{2}{4}a_2 = -\frac{1}{2}A$, $a_6 = \frac{2}{6}a_4 = \frac{1}{2\cdot 3}A$, $a_8 = \frac{2}{8}a_6 = \frac{1}{2\cdot 3\cdot 4}A$, and, in general, $a_{2j} = \frac{1}{j!}A$. The power series solution is $y = A\sum_{j=0}^{\infty} \frac{x^{2j}}{j!}$. This can be recognized as $y = Ae^{x^2}$, the same solution that is obtained by separating variables and integrating.

(b) Substitute to obtain $\sum_{j=1}^{\infty} ja_jx^{j-1} + \sum_{j=0}^{\infty} a_jx^j = 1$. Reindex the second sum, $j \to j-1$, and combine to get $\sum_{j=1}^{\infty}(ja_j + a_{j-1})x^{j-1} = 1$. It follows that $a_1 + a_0 = 1$ and $ja_j + a_{j-1} = 0$ for $j \geq 2$; a_0 can be chosen arbitrarily. To obtain a solution formula, let $a_0 = A$ so $a_1 = 1 - A$. Then write the recursion equation in the form $a_j = -\frac{1}{j}a_{j-1}$ to get $a_2 = -\frac{1}{2}a_1 = -\frac{1}{2}(1-A)$, $a_3 = -\frac{1}{3}a_2 = \frac{1}{2\cdot 3}(1-A)$, $a_4 = -\frac{1}{4}a_3 = -\frac{1}{2\cdot 3\cdot 4}(1-A)$ and, in general, $a_j = -\frac{(-1)^j}{j!}(1-A), j \geq 1$. The power series solution is $y = A - (1-A)\sum_{j=1}^{\infty}(-1)^j\frac{x^j}{j!}$. This can be recognized as $y = 1 - (1-A)e^{-x}$, which is equivalent to the solution obtained by solving the equation as first order linear.

(c,d,e) Problems (c), (d), (e) are very similar to (b).

(f) Substitute to obtain $\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = x^2$. Reindex the second sum, $j \to j-1$, and combine the two sums to get the equation, $\sum_{j=1}^{\infty}(j a_j - a_{j-1})x^{j-1} = x^2$. It follows that $a_1 - a_0 = 2$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 1$, and $j a_j - a_{j-1} = 0$ for $j \geq 4$; a_0 can be chosen arbitrarily.

To obtain a solution formula, let $a_0 = A$ so $a_1 = a_0 = A$, $a_2 = \frac{1}{2}a_1 = \frac{1}{2}A$, and $a_3 = \frac{1}{3}(1+a_2) = \frac{1}{3}(1+\frac{A}{2}) = \frac{1}{3} + \frac{1}{3\cdot 2}A = \frac{1}{3\cdot 2}(2+A)$. Now write the last recursion equation in the form $a_j = \frac{1}{j}a_{j-1}$ to get $a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}(2+A)$, $a_5 = \frac{1}{5}a_4 = \frac{1}{5\cdot 4\cdot 3\cdot 2}(2+A)$ and, in general, $a_j = \frac{1}{j!}(2+A)$ when $j \geq 3$. The power series solution is $y = A + Ax + \frac{1}{2}Ax^2 + (2+A)\sum_{j=3}^{\infty}\frac{x^j}{j!}$. This can be rearranged into $y = Ae^x + 2(e^x - 1 - x - \frac{1}{2}x^2)$ or $y = Ce^x - 2 - 2x - x^2$, where $C = A+2$. This is also the solution that is obtained using the method of undetermined coefficients.

3. Solve $y' = (1-x^2)^{-1/2}$ in two different ways.

<u>Method 1.</u> Using the binomial series:

$$\begin{aligned}(1+x)^p &= 1 + px + \frac{p(p-1)}{1\cdot 2}x^2 + \cdots + \frac{p(p-1)\cdots(p-j+1)}{j!}x^j + \cdots \\ &= 1 + \sum_{j=1}^{\infty}\frac{p(p-1)\cdots(p-j+1)}{j!}x^j\end{aligned}$$

we have

$$\begin{aligned}(1-x^2)^{-1/2} &= 1 + \sum_{j=1}^{\infty}\frac{(-\frac{1}{2})\cdot(-\frac{3}{2})\cdots(-\frac{2j-1}{2}}{j!}(-x^2)^j \\ &= 1 + \sum_{j=1}^{\infty}\frac{1\cdot 3\cdots(2j-1)}{2^j j!}x^{2j}.\end{aligned}$$

Integrate $y' = (1-x^2)^{-1/2}$ term by term to obtain the general solution in power series from: $y = x + \sum_{j=1}^{\infty}\frac{1\cdot 3\cdots(2j-1)}{2^j j!(2j+1)}x^{2j+1} + C$. The arcsine function is obtained by setting $C = 0$ to obtain the solution to the differential equation satisfying $y(0) = 0$:

$$\arcsin x = x + \sum_{j=1}^{\infty}\frac{1\cdot 3\cdots(2j-1)}{2^j j!(2j+1)}x^{2j+1},\ |x| < 1.$$

The formula for $\pi/6$ is obtained by substituting $x = 1/2$. Note that $2^j j! = 2 \cdot 4 \cdots 2j$.

Method 2. Using the power series method to solve $y' = (1-x^2)^{-1/2}$ we could start off by substituting $y = \sum_{j=0}^{\infty} a_j x^j$ into the equation

$$y' = 1 + \sum_{j=1}^{\infty} \frac{1 \cdot 3 \cdots (2j-1)}{2^j j!} x^{2j}$$

to get

$$\sum_{j=1}^{\infty} j a_j x^{j-1} = 1 + \sum_{j=1}^{\infty} \frac{1 \cdot 3 \cdots (2j-1)}{2^j j!} x^{2j}.$$

The reader is invited to finish. Hint. First observe that a_0 can be chosen arbitrarily, let $a_0 = A$. Then let b_j denote the j^{th} coefficient for the series on the right side. The equation $(j+1)a_{j+1} = b_j$, $j \geq 0$, serve as the recursion relations.

5. First solve $y' = x - y$, $y(0) = 0$ by substituting $y = \sum_{j=0}^{\infty} a_j x^j$. Note that $a_0 = y(0) = 0$. Substituting, rearranging, and reindexing produces $\sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=1}^{\infty} a_{j-1} x^{j-1} = x$, so $\sum_{j=1}^{\infty} (j a_j + a_{j-1}) x^{j-1} = x$. Consequently, $a_1 + a_0 = 0$ so $a_1 = 0$ also. For $j = 2$, $2a_2 + a_1 = 1$, so $a_2 = \frac{1}{2}$. Now use $j a_j + a_{j-1} = 0, j \geq 3$, in the form $a_j = -\frac{1}{j} a_{j-1}$ to obtain $a_3 = -\frac{1}{3 \cdot 2}$, $a_4 = \frac{1}{4 \cdot 3 \cdot 2}$, $\cdots$, $a_j = \frac{(-1)^j}{j!}$, $\cdots$. The solution is $y = \sum_{j=2}^{\infty} (-1)^j \frac{x^j}{j!} = e^{-x} - 1 + x$. The algorithm for first order linear equation yields the same solution.

Now solve the IVP using the method of repeated differentiation as described in problem 4(b). We know that $y(0) = 0$. Since $y'(0) = 0 - y(0)$, $y'(0) = 0$ also. For the higher order terms, repeated differentiation yields

$$y' = x - y, y'' = 1 - y', y''' = -y'', \text{and } y^{(j)} = -y^{(j-1)}, j \geq 3.$$

Therefore, $y''(0) = 1 - y'(0) = 1, y'''(0) = -y''(0) = -1, y^{(4)}(0) = -y^{(3)}(0) = 1$, and so on. Consequently, $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$, $a_2 = y''(0)/2! = 1/2$, and $a_j = (-1)^j y^{(j)}(0)/j! = (-1)^j/j!$ for $j \geq 3$, as above.

3.3 Second-Order Linear Equations: Ordinary Points

1. Make sure p and q are real analytic at 0, then substitute $y = \sum_{j=0}^{\infty} a_j x^j$. We work the first problem in detail and summarize the results for the remaining five. In each case $a_0 = A$ and $a_1 = B$.

(a) $p(x) = x$ and $q(x) = 1$. Substitution yields $\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + \sum_{j=0}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$ which, after reindexing the first sum, can be rearranged to $\sum_{j=0}^{\infty}[(j+2)(j+1)a_{j+2} + (j+1)a_j]x^j = 0$. Consequently, $(j+2)(j+1)a_{j+1} + (j+1)a_j = 0$ for all $j \geq 0$. Using the recursion relation in the form $a_{j+2} = -\frac{1}{j+2}a_j$, and starting with $a_0 = A$ we have $a_2 = -\frac{1}{2}A$, $a_4 = \frac{1}{2\cdot 4}A$, and in general, $a_{2j} = \frac{(-1)^j}{2\cdot 4 \cdots (2j)}A$. One solution is $y_1 = A(1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{2\cdot 4\cdots(2j)}) = A\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^j j!}$. Starting with $a_1 = B$ we get $a_3 = -\frac{1}{3}B$, $a_5 = \frac{1}{3\cdot 5}B$, and in general $a_{2j-1} = \frac{(-1)^{j-1}}{1\cdot 3\cdots(2j-1)}B$. Another solution is $y_2 = B\sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^{2j-1}}{1\cdot 3\cdots(2j-1)}$ and the complete general solution, in power series form, is $y = A\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^j j!} + B\sum_{j=1}^{\infty} \frac{(-1)^{j-1}x^{2j-1}}{1\cdot 3\cdots(2j-1)}$.

(b) $p(x) = -1$, $q(x) = x$. Substitute and rearrange to

$$\sum_{j=3}^{\infty}[j(j-1)a_j - (j-1)a_{j-1} + a_{j-3}]x^{j-2} = a_1 - 2a_2.$$

Consequently, $a_2 = \frac{1}{2}B$, and the remaining coefficients can be obtained using the three term relation $a_j = \frac{(j-1)a_{j-1} - a_{j-3}}{j(j-1)}$, $j \geq 3$. Solution:

$$y = A(1 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \cdots) + B(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots).$$

(c) $p(x) = 2x$, $q(x) = -1$. Substitute and rearrange to

$$\sum_{j=1}^{\infty}[(j+2)(j+1)a_{j+2} + (2j-1)a_j]x^j = a_0 - 2a_2 + x.$$

Therefore, $a_2 = \frac{1}{2}A$ and $6a_3 + a_1 = 1$, implying that $a_3 = \frac{1}{6}(1-B)$. When $j \geq 2$, $a_{j+2} = -\frac{2j-1}{(j+2)(j+1)}a_j$. Solution:

$$y = A(1 + \frac{1}{2}x^2 + \cdots) + B(x - \frac{1}{6}x^3 + \cdots) + \frac{1}{6}x^3 - \frac{1}{24}x^5 + \cdots .$$

(d) $p(x) = 1$, $q(x) = -x^2$. Substitute and rearrange to

$$\sum_{j=4}^{\infty}[j(j-1)a_j + (j-1)a_{j-1} - a_{j-4}]x^{j-2} = -2a_2 - a_1 + 1 - (6a_3 + 2a_1)x.$$

Consequently, $a_2 = \frac{1}{2}(1 - B)$ and $a_3 = -\frac{1}{3}B$. When $j \geq 4$, $a_j = \frac{a_{j-4} - (j-1)a_{j-1}}{j(j-1)}$. Solution:

$$y = A(1 + \frac{1}{12}x^4 + \cdots) + B(x - \frac{1}{2}x^2 + \cdots) + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots .$$

(e) $p(x) = x/(1+x^2)$ and $q(x) = 1/(1+x^2)$. Substitute and rearrange to

$$\sum_{j=2}^{\infty}[(j+2)(j+1)a_{j+2} + (j^2+1)a_j]x^j = -(2a_2 + a_0) - (6a_3 + 2a_1)x.$$

Consequently, $a_2 = -\frac{1}{2}A$ and $a_3 = -\frac{1}{3}B$. If $j \geq 2$, then $a_{j+2} = -\frac{j^2+1}{(j+2)(j+1)}a_j$. Solution:

$$y = A(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots) + B(x - \frac{1}{3}x^3 + \frac{1}{6}x^5 + \cdots).$$

(f) $p(x) = 1 + x$, $q(x) = -1$. Substitute and rearrange to

$$\sum_{j=1}^{\infty}[(j+2)(j+1)a_{j+2} + (j+1)a_{j+1} + (j-1)a_j]x^j = a_0 - a_1 + 2a_2.$$

Therefore, $a_2 = \frac{1}{2}(A-B)$, and when $j \geq 1$, $a_{j+2} = -\frac{(j+1)a_{j+1} + (j-1)a_j}{(j+2)(j+1)}$. Solution:

$$y = A(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots) + B(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots).$$

3. Consider $y'' + xy' + y = 0$.

(a) Substitute $y = \sum_{j=0}^{\infty} a_j x^j$, reindex

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j + \sum_{j=1}^{\infty} ja_jx^j + \sum_{j=0}^{\infty} a_jx^j = 0.$$

and rearrange to

$$\sum_{j=1}^{\infty}[(j+2)(j+1)a_{j+2} + (j+1)a_j]x^j = -(2a_2 + a_0).$$

Consequently, $a_2 = -\frac{1}{2}a_0$ and $a_{j+2} = -\frac{1}{j+2}a_j$ for $j \geq 1$. If $a_0 = 1$ and $a_1 = 0$, then $a_j = 0$ if j is odd. Also $a_4 = -\frac{1}{4}a_2 = \frac{1}{2\cdot 4}$, $a_6 = -\frac{1}{6}a_4 = -\frac{1}{2\cdot 4\cdot 6}$, and, in general, $a_{2j} = \frac{(-1)^j}{2\cdot 4\cdots(2j)}$. Thus, one solution is $y_1(x) = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j}}{2^j j!}$. A similar calculation, starting with $a_0 = 0$ and $a_1 = 1$, yields $y_2(x) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}x^{2j-1}}{1\cdot 3\cdot 5\cdots(2j-1)}$ as a second, linearly independent, solution.

(b) The ratio test applied to the series for $y_1(x)$:

$$\lim_{j\to\infty}\left|\frac{x^{2(j+1)}/(2^{j+1}(j+1)!)}{x^{2j}/(2^j j!)}\right| = \lim_{j\to\infty}\frac{x^2}{2(j+1)} = 0,$$

shows that the radius of convergence is ∞. A similar calculation will show that the series for $y_2(x)$ also converges for all x.

(c) Write the series defining y_1 as $\sum_{j=0}^{\infty}\frac{(-x^2/2)^j}{j!}$ to see that $y_1(x) = e^{-x^2/2}$. According to the method developed in Section 2.4 there is a solution y_3 of the form

$$y_1(x)\int\frac{1}{y_1(x)^2}e^{-\int p(x)dx}dx = e^{-x^2/2}\int\frac{1}{e^{-x^2}}e^{-x^2/2}dx.$$

Thus $y_3(x) = e^{-x^2/2}\int e^{x^2/2}dx$. Choosing the antiderivative so its value is 0 at $x = 0$, say $\int_0^x e^{t^2/2}dt$, will make y_3 the solution of the differential equation satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Since y_2 satisfies the same conditions, $y_2 = y_3$.

5. Investigate power series solution to $y'' + (p + 1/2 - x^2/4)y = 0$.

(a) Substitute $y = \sum_{j=0}^{\infty} a_j x^j$. Then reindex and combine to obtain

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)a_{j+2} + (p + \frac{1}{2})a_j - \frac{1}{4}a_{j-2} \right] x^j = 0,$$

where, by convention, $a_{-2} = a_{-1} = 0$.

(b) If $y = we^{-x^2/4}$, then

$$y' = w'e^{-x^2/4} - \frac{x}{2}we^{-x^2/4}$$
$$y'' = w''e^{-x^2/4} - xw'e^{-x^2/4} - \frac{1}{2}we^{-x^2/4} + \frac{x^2}{4}we^{-x^2/4}.$$

Substitute and simplify, eventually canceling the exponential terms, to obtain the desired equation: $w'' - xw' + pw = 0$.

(c) Substitute $w = \sum_{j=0}^{\infty} b_j x^j$. Reindex and rearrange to

$$\sum_{j=0}^{\infty} [(j+2)(j+1)b_{j+2} + (p-j)b_j]x^j = 0.$$

The two-term recursion formula is $b_{j+2} = -\frac{p-j}{(j+2)(j+1)}b_j, j \geq 0$. Starting with $b_0 = A$, $b_2 = -\frac{p}{2\cdot 1}A$, $b_4 = -\frac{p-2}{4\cdot 3}b_2 = \frac{p(p-2)}{4!}A$, $b_6 = -\frac{p-4}{6\cdot 5}b_4 = -\frac{p(p-2)(p-4)}{6!}A$, and so on. One solution is

$$y_1 = A\left(1 - \frac{p}{2!}x^2 + \frac{p(p-2)}{4!}x^4 - \cdots\right).$$

Starting with $b_1 = B$, the second solution is

$$y_2 = B\left(x - \frac{p-1}{3!}x^3 + \frac{(p-1)(p-3)}{5!}x^5 - \cdots\right).$$

7. Chebyshev's equation: $(1-x^2)y'' - xy' + p^2y = 0$.

(a) Substitute $y = \sum_{j=0}^{\infty} a_j x^j$. Reindex and rearrange to

$$\sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} + (p^2 - j^2)a_j]x^j = 0.$$

Using the recursion relation $a_{j+2} = -\frac{p^2-j^2}{(j+2)(j+1)}a_j$ the general solution is

$$\begin{aligned} y = & A\left(1 - \frac{p^2}{2!}x^2 + \frac{p^2(p^2-2^2)}{4!}x^4 - \cdots\right) + \\ & B\left(x - \frac{p^2-1}{3!}x^3 + \frac{(p^2-1)(p^2-3^2)}{5!}x^5 - \cdots\right). \end{aligned}$$

(b) If p is an *even* integer, then the first series terminates in a polynomial of degree p. If p is an *odd* integer, then the second series terminates in a polynomial of degree p.

3.4 Regular Singular Points

1. Locate and classify the singular points.

(a) $p(x) = -\frac{2}{x^3}$, $q(x) = \frac{3}{x^2(x-1)}$. The singular points are 0 and 1, 1 is regular and 0 is not.

(b) $p(x) = \frac{1}{x(x+1)}$, $q(x) = \frac{2}{x^2(x^2-1)}$. The singular points are 0 and ± 1, all three are regular.

(c) $p(x) = -\frac{x-2}{x^2}$, $q(x) = 0$. The singular point is 0 which is not regular.

(d) $p(x) = -\frac{x+1}{x(3x+1)}$, $q(x) = \frac{2}{x(3x+1)}$. The singular points are 0 and $-1/3$, both are regular.

3. Find the indicial equation and its roots. Note that if the equation is in the form $y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$, then the indicial equation is $m(m-1) + p(0)m + q(0) = 0$.

(a) $p(x) = \frac{\cos 2x - 1}{x^2}$, $q(x) = 2$, $p(0) = -2$ (use p's Taylor series) and $q(0) = 2$ so the indicial equation is $m(m-1) - 2m + 2 = 0$. The roots are $m_{1,2} = 1, 2$.

(b) $p(x) = \frac{2x^3-5}{4}$, $q(x) = \frac{3x^2+2}{4}$. $p(0) = -5/4$ and $q(0) = 1/2$ so the indicial equation is $m(m-1) - \frac{5}{4}m + \frac{1}{2} = 0$. The roots are $m_{1,2} = \frac{1}{4}, 2$.

(c) $p(x) = 3$, $q(x) = 4x$. $p(0) = 3$ and $q(0) = 0$ so the indicial equation is $m(m-1) - 4m + 3 = 0$. The roots are $m_{1,2} = 5/2 \pm \sqrt{13}/2$.

5. Write the equation in the form $y'' + \frac{1}{x}y' + \frac{x^2}{x^2}y = 0$ to see that the indicial equation is $m(m-1) + m = 0$ so the indicial roots are $m_{1,2} = 0, 0$. This suggests that there is an ordinary power series solution: $y = \sum_{j=0}^{\infty} a_j x^j$. Substitute, reindex, and rearrange to $\sum_{j=1}^{\infty}[j^2 a_j + a_{j-2}]x^j = 0$ where we use the convention that $a_{-1} = 0$. Consequently, a_0 can be chosen arbitrarily, $a_1 = 0$, and $a_j = -\frac{1}{j^2}a_{j-2}$ for all $j \geq 2$. Start with $a_0 = 1$ to obtain the series displayed in the text.

7. Frobenius solutions for $y'' + \frac{p}{x^b}y' + \frac{q}{x^c} = 0$, p and q nonzero real numbers.

(a) Consider $y'' + \frac{p}{x^2}y' + \frac{q}{x^3} = 0$. Write it in the form $x^3y'' + pxy' + qy = 0$ and substitute $y = \sum_{j=0}^{\infty} a_j x^{m+j}$. Since

$$\begin{aligned} qy &= q(a_0x^m + a_1x^{m+1} + \cdots) \\ pxy' &= p(a_0mx^m + a_1(m+1)x^{m+1} + \cdots) \\ x^3y'' &= a_0m(m-1)x^{m-1} + a_1(m+1)mx^{m+2} + \cdots \end{aligned}$$

we obtain

$$(pm+q)a_0x^m + [m(m-1)a_0 + (p(m+1)+q)a_1]x^{m+1} + \cdots = 0.$$

Therefore, a_0 can be chosen arbitrarily provided $pm + q = 0$. This is the indicial equation for this case and there is one possible value for the exponent, $m = -q/p$.

(b) Consider now the general case $y'' + px^{-b}y' + qx^{-c}y = 0$. THe same substitution that was made in part (a) now entails

$$\begin{aligned} qx^{-c}y &= q(a_0x^{m-c} + a_1x^{m+1-c} + \cdots) \\ px^{-b}y' &= p(a_0mx^{m-1-b} + a_1(m+1)x^{m-b} + \cdots) \\ y'' &= a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + \cdots \end{aligned}$$

Add these up, and cancel x^m from each term, to obtain the following equation

$$\begin{aligned} &qa_0x^{-c} + pma_0x^{-b-1} + a_0m(m-1)x^{-2} + \\ &qa_1x^{1-c} + p(m+1)a_1x^{-b} + a_1(m+1)mx^{-1} + \cdots = 0. \end{aligned} \tag{3.1}$$

If $b = 1$ and $c \leq 2$, then after multiplying Equation **??** by x^2 it can be rearranged to

$$[m(m-1) + pm]a_0 + qa_0x^{2-c} + \text{higher order terms} = 0.$$

Thus we can hope for two values of m for Frobenius solutions.

If $b = 1$ and $c > 2$, then it is clear from the first term in Equation **??** that $a_0 = 0$ and there is no hope for a Frobenius solution.

If $b \neq 1$ and $c = b + 1$, then there will be one value of m that might yield a Frobenius solution, as in part (a). Otherwise, and still assuming $b \neq 1$, there will be no Frobenius solutions because the three leading terms in Equation **??** force $a_0 = 0$.

3.5 More on Regular Singular Points

1. Since $p(x) = -3$ and $q(x) = 4x+4$, the indicial equation is $m(m-1) - 3m + 4 = 0$. The exponents are $m_{1,2} = 2, 2$ and there is a solution of the form $y = x^2 \sum_{j=0}^{\infty} a_j x^j$. Substitute, reindex, and rearrange to

$$\sum_{j=0}^{\infty} (j^2 a_j + 4a_{j-1})x^j = 0.$$

We are using the convention that $a_{-1} = 0$. Consequently, a_0 can be chosen arbitrarily, say $a_0 = 1$, and the rest of the coefficients are found using the relation $a_j = -\frac{4}{j^2}a_{j-1}, j \geq 1$. The Frobenius solution is $y = x^2 \sum_{j=0}^{\infty}(-1)^n \frac{4^n}{(n!)^2}x^j$.

3. Find two independent Frobenius solutions.

(a) $p(x) = 2$ and $q(x) = x^2$. The indicial equation is $M^2+m = 0$ so the exponents are $m_{1,2} = 0, -1$. We will substitute $y = \sum_{j=0}^{\infty} a_j x^{m+j}$ with the hope that both solutions can be obtained using $m_2 = -1$. IF that is not the case, then we will at least have the solution $y+1$ corresponding to $m_1 = 0$ and can build an independent solution using the integral formula for a second independent solution.

Substitute, reindex, and simplify to obtain the following sum

$$\sum_{j=0} [(m+j)(m+j+1)a_j + a_{j-2}]x^{j-1} = 0.$$

We are using the convention that a_{-1} and a_{-2} are both zero. The recursion relations start with

$$j = 0 \implies m(m+1)a_0 = 0 \tag{3.2}$$
$$j = 1 \implies (m+1)(m+2)a_1 = 0 \tag{3.3}$$

Equation **??** confirms that $m = 0$ and $m = -1$ are the exponents. Either value for m will allow us to choose a_0 arbitrarily. Moreover, if we choose $m = -1$, then Equation **??** implies that a_1 can also be chosen arbitrarily. We get two independent Frobenius series solutions using $m = -1$.

> **We got lucky.** This is pure luck. Sometimes it happens, sometimes it does not. Unfortunately, there is no way to tell in advance that it will happen.

In view of what we have just seen, we let $m = -1$ and use the recursion relation $a_j = -\frac{1}{j(j-1)}a_{j-2}, j \geq 2$, to determine the rest of the coefficients. Independent solutions are generated by first starting out with $a_0 = 1$ and $a_1 = 0$, an then starting out with $a_0 = 0$ and $a_1 = 1$.

$\underline{a_0 = 1, a_1 = 0}$

In this case, $a_2 = -\frac{1}{2\cdot 1}$, $a_4 = \frac{1}{4\cdot 3\cdot 2\cdot 1}$, and, in general, $a_{2j} = \frac{(-1)^j}{(2j)!}$. The solution is $y_1 = x^{-1}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}x^{2j} = \frac{\cos x}{x}$.

$\underline{a_0 = 0, a_1 = 1}$

In this case, $a_3 = -\frac{1}{3\cdot 2}$, $a_5 = \frac{1}{5\cdot 4\cdot 3\cdot 2}$ and, in general, $a_{2j+1} = \frac{(-1)^j}{(2j+1)!}$. The solution is $y_2 = x^{-1}\sum_{j=0}^{\infty}\frac{(-1)^J}{(2j+1)!}x^{2j} = \frac{\sin x}{x}$.

(b) $p(x) = -x$, $q(x) = x^2 - 2$. The indicial equation is $m(m-1) - 2 = 0$ and $m_{1,2} = 2, -1$. Substitute and simplify to obtain

$$\sum_{j=0}^{\infty}[(m+j-2)(m+j+1)a_j - (m+j-1)a_{j-1} + a_{j-2}]x^j = 0.$$

This is correct under the assumption that a_{-1} and a_{-2} are both zero. The exponents are obtained from the coefficients when $j = 0$: $(m-2)(m+1)a_0 = 0$. This confirms that the exponents are 2 and -1 as expected. Let's investigate what happens when we set $m = -1$. The recursion relation in this is

$$j(j-3)a_j - (j-2)a_{j-1} + a_{j-2} = 0,$$

and the coefficient calculations for $j = 0, 1, 2, 3$ go like this:

$$\begin{aligned}
j = 0 &\implies 0 \cdot a_0 = 0, && \text{so } a_0 \text{ is arbitrary} \\
j = 1 &\implies -2 \cdot a_1 + a_0 = 0, && \text{so } a_1 = \frac{1}{2}a_0 \\
j = 2 &\implies -2 \cdot a_2 + a_0 = 0, && \text{so } a_2 = \frac{1}{2}a_0 \\
j = 3 &\implies 0 \cdot a_3 - a_2 + a_1 = 0, && \text{so } a_3 \text{ is arbitrary}
\end{aligned}$$

We got lucky again. Observe that once $j > 3$ the recursion relation can be rearranged to $a_j = \frac{(j-2)a_{j-1} - a_{j-2}}{j(j-3)}$ allowing us to calculate the remaining coefficients in terms of a_0 and a_3. Independent solutions y_1 and y_2 arise from $a_0 = 1$, $a_3 = 0$ and $a_0 = 0$, $a_3 = 1$ respectively.

$$\begin{aligned}
y_1 &= x^{-1}(1 + \frac{x}{2} + \frac{x^2}{2} + \cdots) \\
y_2 &= x^{-1}(x^3 + \frac{x^4}{2} + \frac{x^5}{20} + \cdots).
\end{aligned}$$

(c) $p(x) = -1$, $q(x) = 4x^4$. The indicial equation is $m(m-1) - m = 0$ so the exponents are $m_{1,2} = 0, 2$. Substitute and simplify to obtain

$$\sum_{j=0}^{\infty} [(m+j)(m+j-2)a_j + 4a_{j-4}]x^{j-1} = 0.$$

The coefficients with negative indices are set to 0. The exponents are confirmed using the $j = 0$ relation: $m(m-2)a_0 = 0$. They are 0 and 2 as we expect.

Let's try out lick again. Set $m = 0$ and investigate the first few recursion relations. We already know a_0 is arbitrary and the recursion relations are derived from the equation $j(j-2)a_j +$

$4a_{j-4} = 0$.

$$\begin{aligned} j = 1 &\implies -a_1 = 0, && \text{so } a_1 = 0 \\ j = 2 &\implies 0 \cdot a_2 = 0, && \text{so} a_2 \text{ is arbitrary} \\ j = 3 &\implies a_j = -\frac{4}{j(j-2)a_{j-4}}, && \text{to calculate the rest} \end{aligned}$$

Once more we have two arbitrary constants. Starting from $a_0 = 1$ and $a_2 = 0$, $a_1 = 0$ and $a_3 = 0$ also, and the recursion relation implies that all coefficients that are not multiples of 4 must be 0. Since $a_{4j} = -\frac{1}{2j(2j-1)}a_{4j-4}$, it is easily seen that $a_{4j} = \frac{(-1)^j}{(2j)!}$, and one solution is $y_1 = \sum_{j=0}^{\infty}(-1)^j \frac{x^{4j}}{(2j)!} = \cos(x^2)$.

The solution derived from $a_0 = 0$ and $a_2 = 1$ is $y_2 = \sin(x^2)$ (verify).

5. The equation $y'' - \frac{1}{3(x+1)}y' - \frac{1}{3(x+1)^2}y = 0$ and $x = -1$ is clearly a singular point. Since $(x+1)p(x) = -1/3$ and $(x+1)^2 q(x) = -1/3$, -1 is a regular singular point and the differential equation will have at least one solution of the form $y = (x+1)^m \sum_{j=0}^{\infty} a_j(x+1)^j$, a_0 arbitrary.

To facilitate the computation of the coefficients it is convenient to make a simple change of independent variable: $t = x + 1$. Then $y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} = \dot{y}$, and a similar computation will show $y'' = \ddot{y}$. Therefore, the differential equation becomes $3t^2\ddot{y} - t\dot{y} - y = 0$ and we may substitute $y = t^m \sum_{j=0}^{\infty} a_j t^j$. Because it is an Euler equidimensional equation, the substitution $y = t^m$ works just as well. The simplified result is the equation $3m(m-1) - m - 1 = 0$ or $3m^2 - 4m - 1 = 0$ so $m_{1,2} = \frac{2}{3} \pm \frac{\sqrt{7}}{3}$. Consequently, the general solution to the original equation is

$$y = (x+1)^{2/3}\left(c_1(x+1)^{\sqrt{7}/3} + c_2(x+1)^{-\sqrt{7}/3}\right).$$

7. Since $p(x) = 1$ and $q(x) = x^2 - 1/4$, the indicial equation is $m(m-1) + m - 1/4 = 0$ and the exponents are $m_{1,2} = \pm 1/2$. Thus $m_1 - m_2 = 1$. Make the substitution $y = x^m \sum_{j=0}^{\infty} a_j x^j$ into the equation and simplify to

$$\sum_{j=0}^{\infty}[((m+j)^2 - \frac{1}{4})a_j + a_{j-2}]x^j = 0.$$

As usual, $a_{-1} = 0$ and $a_{-2} = 0$. Let $m = -1/2$ to obtain the recursion relation

$$j(j-1)a_j + a_{j-2} = 0, j \geq 0.$$

Then $0 \cdot a_0 = 0$ and $0 \cdot a_1 = 0$ imply that both a_0 and a_1 can be chosen arbitrarily. The remaining coefficients are found using the relation $a_j = -\frac{1}{j(j-1)} a_{j-2}, j \geq 2$.

$$a_0 = 1, a_1 = 0 \implies y_1 = x^{-1/2}(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) = x^{-1/2} \cos x$$
$$a_0 = 0, a_1 = 1 \implies y_2 = x^{-1/2}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) = x^{-1/2} \sin x.$$

3.6 Gauss's Hypergeometric Equation

1. Verify the identity.

(a) $F(-p, b, b, -x) = 1 + \sum_{j=1}^{\infty} \frac{-p(-p+1)\cdots(-p+j-1)}{j!}(-x)^j$. Therefore,

$$F(-p, b, b, -x) = 1 + \sum_{j=1}^{\infty} \frac{p(p-1)\cdots(p-j+1)}{j!} x^j = (1+x)^p.$$

(b) See the last displayed equation in this section.

(c) $xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{3}, x^2) = x + \sum_{j=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+j-1)\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+j-1)}{j!\frac{3}{2}(\frac{3}{2}+1)\cdots(\frac{3}{2}+j-1)} x^{2j+1}$. Cancel terms and simplify to obtain

$$xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) = x + \sum_{j=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2j-1)}{2^j j!(2j+1)} x^{2j+1} = \arcsin x.$$

(d) $xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = x + x\sum_{j=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+j-1)1\cdot 2\cdots j}{j!\frac{3}{2}(\frac{3}{2}+1)\cdots(\frac{3}{2}+j-1)}(-x^2)^j$. Cancel terms and simplify to

$$xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = x + \sum_{j=1}^{\infty} \frac{(-1)^j}{2j+1} x^{2j+1} = \arctan x.$$

(e) $F(a,b,a,\frac{x}{b} = 1+\sum_{j=1}^{\infty}\frac{b(b+1)\cdots(b+j-1)}{j!}\cdot\frac{x^j}{b^j} = 1+\sum_{j=1}^{\infty}\frac{1(1+\frac{1}{b})\cdots(1+\frac{j-1}{b})}{j!}x^j$. Allow $b\to\infty$, term by term, to obtain

$$F(a,b,a,\frac{x}{b}) = 1+\sum_{j=1}^{\infty}\frac{x^j}{j!} = e^x.$$

(f) $xF(a,a,\frac{3}{2},\frac{-x^2}{4a^2}) = x + x\sum_{j=1}^{\infty}\frac{a^2(a+1)^2\cdots(a+j-1)^2}{j!\frac{3}{2}(\frac{3}{2}+1)\cdots(\frac{3}{2}+j-1)}\cdot\frac{(-x^2)^j}{4^ja^{2j}}$. Distribute a^{2j} into the numerator and simplify the denominator to obtain

$$xF(a,a,\frac{3}{2},\frac{-x^2}{4a^2} = x+\sum_{j=1}^{\infty}\frac{1(1+\frac{1}{a})^2\cdots(1+\frac{j-1}{a})^2}{j!\cdot 2^j\cdot 3\cdot 5\cdots(2j+1)}(-1)^jx^{2j+1}.$$

Now let $a\to\infty$ term by term, and use $j!\cdot 2^j = 2\cdot 4\cdots(2j)$, to end up with

$$xF(a,a,\frac{3}{2},\frac{-x^2}{4a^2)}) = x+\sum_{j=1}^{\infty}\frac{(-1)^j}{(2j+1)!}x^{2j+1} = \sin x.$$

(g) Just like (f).

3. Make the substitution $t = \frac{1-x}{2}$. This implies that $y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{1}{2}\dot{y}$. Similarly, $y'' = \frac{1}{4}\ddot{y}$, and the differential equation is converted into

$$4t(1-t)\cdot\frac{1}{4}\cdot\ddot{y} + (2t-1)\cdot(-\frac{1}{2})\cdot\dot{y} + p^2y = 0.$$

Simplify to $t(1-t)\ddot{y} + (\frac{1}{2}-t)\dot{y} + p^2y = 0$, which is easily recognized as hypergeometric with $c=\frac{1}{2}$, $a=p$, and $b=-p$. Consequently, the general solution, in terms of t, is $y = c_1F(p,-p,1/2,t) + c_2t^{1/2}F(p+1/2,-p+1/2,3/2,t)$. The solution in terms of x is obtained by replacing t with $(1-x)/2$.

5. If $t=e^x$, then $y' = t\dot{y}$ and $y'' = t^2\ddot{y} + t\dot{y}$. After substitution and simplification, the differential equation for $y(t)$ is hypergeometric of the form $t(1-t)\ddot{y} + (3/2-t)\dot{y} + y = 0$. Clearly $c = 3/2$ and $a=-b=1$. The formula for the general solution near $t=1$ yields

$$y = c_1F(1,-1,-1/2,1-t) + c_2(1-t)^{3/2}F(5/2,1/2,5/2,1-5).$$

Replace t with e^x to obtain the general solution of the original equation near $x=0$.

Chapter 4

Numerical Methods

4.1 Introductory Remarks

4.2 The Method of Euler

Use Euler's method, $h = 0.1, 0.05, 0.01$, to estimate the solution at $x = 1$.

1. $y' = 2x + 2y$, $y(0) = 1$; solution: $y(x) = -\frac{1}{2} - x + \frac{3}{2}e^{2x}, y(1) = 9.58358$.

Table 4.1: 5.2.1

h	Approx	Error	% Error
0.1	7.78760	1.79600	18.7
0.05	8.59125	0.9924	10.3
0.01	9.36697	0.2166	2.3

3. $y' = e^y$, $y(0) = 0$; solution: $y(x) = -\ln(1 - x)$, $\lim_{x \to 1^-} y(x) = +\infty$.

Table 4.2: 5.2.3

h	Approx	Error	% Error
0.1	2.27337	∞	
0.05	2.74934	∞	
0.01	3.95946	∞	

5. $y' = (x+y-1)^2$, $y(0) = 0$; solution: $y(x) = 1-x-\cot(x+\pi/4), y(1) = 0.217960$.

Table 4.3: 5.2.5

h	Approx	Error	% Error
0.1	0.254474	-0.036514	-17.6
0.05	0.235521	-0.017561	-8.45
0.01	0.221369	-0.003409	-1.64

7. The solution values should increase steadily towards the stable equilibrium, $\phi_2 \equiv 1$. F

4.3 The Error Term

1. $y' = 2x + 2y, y(0) = 1$; solution: $y(x) = -\frac{1}{2} - x + \frac{3}{2}e^{2x}$. Therefore, $y''(x) = 6e^{2x}$ and the aggregate error at $x = 1$ is bounded by $3e^2 h$.

Table 4.4: 5.3.1

h	Error Bound	Actual Error
0.2	4.4334	3.0162
0.1	2.2167	1.7960

3. $y' = e^y, y(0) = 0$; solution: $y(x) = -\ln(1-x)$. Therefore, $y''(x) = 1/(1-x)^2$ and the aggregate error at $x = 1$ is undefined. We examine the situation for $x = 0.8$ instead where the aggregate error is bounded by $0.8 \cdot \frac{25 \cdot h}{2}$.

Table 4.5: 5.3.3

h	Error Bound	Actual Error
0.2	2.0	0.42727
0.1	1.0	0.27318

5. $y' = (x + y - 1)^2, y(0) = 0$; solution: $y(x) = 1 - x - \cot(x + \pi/4)$. Therefore, $y''(x) = -2\csc^2(x+\pi/4)\cot(x+\pi/4)$. The maximum value of $|y''(x)|$ is r, attained at $x = 0$. Therefore, the aggregate error at $x = 1$ is bounded by $4h/2$.

Table 4.6: 5.3.5

h	Error Bound	Actual Error
0.2	0.4	-0.07966
0.1	0.2	-0.03652

4.4 An Improved Euler Method

1. $y' = 2x + 2y, y(0) = 1$; solution: $y(x) = -\frac{1}{2} - x + \frac{3}{2}e^{2x}, y(1) = 9.58360$.

Table 4.7: 5.4.1

h	Improved Euler Approx	Error	%Error
0.1	9.45695	0.1266	1.321
0.05	9.54935	0.0342	0.357
0.01	9.58213	0.0015	0.015

3. $y' = e^y, y(0) = 0$; solution: $y(x) = -\ln(1 - x)$, $\lim_{x\to 1^-} y(x) = +\infty$.

Table 4.8: 5.4.3

h	Improved Euler Approx	Error	%Error
0.1	3.92301	∞	
0.05	4.60207	∞	
0.01	6.20029	∞	

5. $y' = (x+y-1)^2, y(0) = 0$; solution: $y(x) = 1-x-\cot(x+\pi/4), y(1) = 0.217960$.

Table 4.9: 5.4.5

h	Improved Euler Approx	Error	%Error
0.1	0.218698	-0.00074	-0.340
0.05	0.218123	-0.00016	-0.076
0.01	0.217964	- 0.0000061	-0.0028

4.5 The Runge–Kutta Method

1. $y' = 2x + 2y, y(0) = 1$; solution: $y(x) = -\frac{1}{2} - x + \frac{3}{2}e^{2x}, y(1) = 9.58358$.

Table 4.10: 5.5.1

h	RK Approx	Error	%Error
0.1	9.58333	2.61161e-05	0.00261161
0.05	9.58357	1.77359e-06	0.000177359
0.01	9.58358	3.03308e-09	3.03308e-07

3. $y' = e^y, y(0) = 0$; solution: $y(x) = -\ln(1 - x)$, $\lim_{x \to 1^-} y(x) = +\infty$.

Table 4.11: 5.5.3

h	RK Approx	Error	%Error
0.1	5.40911	∞	
0.05	6.10227	∞	
0.01	7.71171	∞	

5. $y' = (x + y - 1)^2, y(0) = 0$; solution: $y(x) = 1 - x - \cot(x + \pi/4), y(1) =$ 0.217958098.

Table 4.12: 5.5.5

h	RK Approx	Error	%Error
0.1	0.2179592	-0.0000011	-0.00050
0.05	0.2179581	-0.000000049	-0.000022
0.01	0.2179580	-0.0000000004	-0.00000018

Chapter 5

Fourier Series: Basic Concepts

5.1 Fourier Coefficients

1. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi dx = 3\pi/2$; $a_j = \frac{1}{\pi}\int_{-\pi} \pi/2\pi \cos(jx)dx = \frac{1}{j}\sin(\frac{j\pi}{2})$; $b_j = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi \sin(jx)dx = \frac{1}{j}((-1)^j - \cos(\frac{j\pi}{2}))$.

3. $a_0 = \frac{1}{\pi}\int_0^\pi \sin x dx = \frac{2}{\pi}$; $a_j = \frac{1}{\pi}\int_0^\pi \sin x \cos jx dx = \frac{1+(-1)^j}{\pi(j^2-1)}, j > 1$; $a_1 = 0$; $b_j = \frac{1}{\pi}\sin x \sin jx dx = 0, j > 1$; $b_1 = 1/2$.

5. Consider alternative lines of thought. In each case the terms in the Fourier series are the same terms that define the function.

7. Let f be the function in Exercise 2 and g and h be the functions in Exercise 1 and Example 6.1.2 respectively. Draw their graphs and observe that $f(x) = \frac{1}{\pi}(g(x) - (\pi - h(x)) = \frac{1}{\pi}(g(x) + h(x)) - 1$. It follows that the Fourier coefficients satisfy similar relationships. For example, if f_0, g_0, and h_0 denote the constants in the three Fourier series, then $f_0 = \frac{1}{\pi}(g_0 + h_0) - 1$. On the other hand, f_1, g_1, and h_1 denote the first cosine coefficient of each of the three Fourier series, then $f_1 = \frac{1}{\pi}(g_1 + h_1)$, and so on.

5.2 Some Remarks about Convergence

1. Exercise.

3. (a) $a_0 = \frac{1}{\pi}\int_0^\pi x^2 dx = \pi^2/3$; $a_j = \frac{1}{\pi}x^2 \cos jx dx = (-1)^j \frac{2}{j^2}$; $b_j = \frac{1}{\pi}\int_0^\pi x^2 \sin jx dx = 2 \cdot \frac{(-1)^j - 1}{\pi \cdot j^3} + (-1)^j \frac{\pi}{j}$ (integration by parts, or use a table of integrals).

(b) Exercise.

(c) If $x = 0$, then the series converges to 0 so $0 = \pi^2/6 + 2\sum_{j=1}^\infty \frac{(-1)^j}{j^2}$. Consequently, $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

If $x = \pi$, then the series converges to $\pi^2/2$ so $\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{j=1}^\infty \frac{(-1)^{2j}}{j^2}$. Consequently, $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$.

(d) Use Exercise 2 and the first sum in (c) to conclude that $\sum_j \frac{1}{(2j)^2} = \frac{\pi^2}{8} - \frac{\pi^2}{12} = \frac{\pi^2}{24}$.

5. (a) $a_0 = \frac{1}{\pi}\int_{-\pi}^\pi e^x dx = \frac{e^\pi - e^{-\pi}}{\pi} = \frac{2\sinh\pi}{\pi}$; $a_j = \frac{1}{\pi}e^x \cos jx dx = \frac{2\sinh\pi}{\pi} \cdot \frac{(-1)^j}{j^2+1}$; $b_j = \frac{1}{\pi}\int_{-\pi}^\pi e^x \sin jx dx = -\frac{2\sinh\pi}{\pi} \cdot \frac{(-1)^j j}{j^2+1}$ (integration by parts, or use a table of integrals).

(b) Exercise.

(c) At $x = \pi$ the series converges to $\frac{e^\pi + e^{-\pi}}{2} = \cosh\pi$. Therefore, $\cosh x = \frac{\sinh x}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{j=1}^\infty \frac{(-1)^{2j}}{j^2+1}$, and a little algebra will produce the first formula.

At $x = 0$ the series converges to 1 so $1 = \frac{\sinh}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{j=1}^\infty \frac{(-1)^j}{j^2+1}$, and a little more algebra will produce the second formula.

7. Let x be an interior point where f is not continuous. Since f is piecewise monotone, it muse be monotone to the left of x, for example monotone increasing is an open interval (a, x). If s is the least upper bound of the set $S = \{f(t) : a < t < x\}$, then $\lim_{t \to x^-} f(t) = s$.

5.3 Even and Odd Functions

1. Determine whether the each function is odd or even or neither.

3. Assume f is even. Then $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$. Make

Functions	Parity	Reason
$x^5 \sin 2x$	even	odd×odd
$x^2 \sin 2x$	odd	even×even
e^x	neither	
$(\sin x)^3$	odd	odd$^{\text{odd}}$
$\sin x^2$	even	f(even)
$\cos(x + x^3)$	even	f_{even}(odd)
$x + x^2 + x^3$	neither *odd* + *even*	
$\ln \frac{1-x}{1+x}$	odd	$f(-x) = \ln \frac{1-x}{1+x} = -f(x)$

the change-of-variable $x = -u$ in the first integral to obtain

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= \int_{a}^{0} f(-u)(-du) + \int_{0}^{a} f(x)dx \\ &= \int_{a}^{a} f(u)du + \int_{0}^{a} f(x)dx \\ &= 2\int_{0}^{a} f(x)dx.\end{aligned}$$

The verification of formula (4) is quite similar.

5. The function is even. $a_0 = \frac{2}{\pi}\int_0^\pi \cos x/2dx = \frac{4}{\pi}$; $a_j = \frac{2}{\pi}\int_0^\pi \cos\frac{x}{2}\cos jxdx = \frac{4}{\pi}\cdot\frac{(-1)^{j+1}}{4j^2-1}$.

7. (a) Direct computation. The function is even and $a_0 = 0$. $a_j = \frac{2}{\pi}\int_0^\pi(-x+\frac{\pi}{2})\cos jxdx = \frac{2}{\pi}\cdot\frac{1+(-1)^{j+1}}{j^2}$. Note that the even coefficients are 0.

$$f(x) = \frac{4}{\pi}\int_{j=1}^{\infty}\frac{\cos(2j-1)x}{(2j-1)^2}, -\pi \le x \le \pi.$$

(b) Observe that $f(x) = \frac{\pi}{2} - |x|$.

9. The average value of f is clearly equal to $\pi/4$ (so $a_0 = \pi/2$). Moreover, $a_j = \frac{2}{\pi}(\int_0^{\pi/2} x\cos jxdx + \int_{\pi/2}^{\pi}(\pi - x)\cos jxdx)$. These coefficients evaluate as follows.

$$a_j = \frac{2}{\pi j^2}\cdot\left(2\cos(\frac{j\pi}{2} - (1+(-1)^j)\right) = \begin{cases} 0, & j = 1,3,5,\cdots \\ -\frac{8}{\pi j^2}, & j = 2,6,10,\cdots \\ 0, & j = 4,8,12,\cdots \end{cases}$$

Note that the indices for the nonzero coefficients are of the form $2(2j-1)$ and $a_{2(2j-1)} = -\frac{8}{\pi(2(2j-1))^2} = -\frac{2}{\pi}\cdot\frac{1}{(2j-1)^2}$.

11. (a) $a_0 = \frac{2}{\pi}\int_0^\pi x^3 dx = \pi^3/2$; $a_j = \frac{2}{\pi}\int_0^\pi x^3 \cos jx dx = \frac{12}{\pi}\cdot\frac{1+(-1)^{j+1}}{j^4} + 6\pi\frac{(-1)^j}{j^2}$.

(b) i. Substitute $x = 0$ and $x = \pi/2$ into the Fourier series for x^3 to obtain the following two equations.

$$0 = \frac{\pi^3}{4} + 6\pi\sum_{j=1}^{\infty}\frac{(-1)^j}{j^2} + \frac{24}{\pi}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^4} \tag{5.1}$$

$$\frac{\pi^3}{8} = \frac{\pi^3}{4} + 6\pi\sum_{j=1}^{\infty} +6\pi\sum_{j=1}^{\infty}\frac{(-1)^j}{(2j)^2} \tag{5.2}$$

Use Equation **??** to calculate $\sum_{j=1}^{\infty}\frac{(-1)^j}{j^2}$. Then substitute the value into Equation **??** to obtain the desired result.

ii. Let $S = \sum_{j=1}^{\infty}\frac{1}{j^4}$. Then $S = \sum_{j=1}^{\infty}\frac{1}{(2j)^4} + \sum_{j=1}^{\infty}\frac{1}{(2j-1)^4} = \frac{1}{16}S + \frac{\pi^4}{96}$. Solve for S.

13. The identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ can be used to evaluate the integrals for the Fourier series coefficients for $\sin^2 x$. Of course, they will evaluate to the coefficients that appear in the identity so the identity and the Fourier series are one in the same.

15. These identities can be established by appealing to the complex formulas for the sine and cosine functions: $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$. FOr example,

$$\begin{aligned}
\sin^3 x &= -\frac{1}{8i}(e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix})\\
&= -\frac{1}{8i}(e^{3ix} - e^{-3ix} - 3e^{ix} + 3e^{-ix})\\
&= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.
\end{aligned}$$

5.4 Fourier Series on Arbitrary Intervals

1. Calculate the Fourier series for the given function.

(a) $L = 1$. The function is odd so $a_j = 0$, $b_j = \frac{2}{1-0}\int_0^1 x \sin n\pi x dx = \frac{2}{\pi} \cdot \frac{(-1)^{j+1}}{j}$; $x = \frac{2}{\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\sin j\pi x}{j}$, $-1 < x < 1$.

(b) $L = 2$. The function is odd so $a_j = 0$. $b_j = \frac{2}{2}\int_0^2 \sin x \sin(j\pi x/2)dx = 2\pi \sin 2 \cdot (-1)^{j+1}\frac{j}{\pi^2 j^2 - 4}$.

$$\sin x = 2\pi \sin 2 \sum_{j=1}^{\infty}(-1)^{j+1}\frac{j}{\pi^2 j^2 - 4}\sin(\frac{j\pi x}{2}), -2 < x < 2.$$

(c) $L = 3$. $a_0 = \frac{1}{3}\int_{-3}^{3} e^x dx = \frac{2}{3}\sinh(3)$; $a_j = \frac{1}{3}\int_{-3}^{3} e^x \cos(j\pi x/3)dx = 6\sinh(3) \cdot \frac{(-1)^j}{\pi^2 j^2 + 9}$; $b_j = \frac{1}{3}\int_{-3}^{3} e^x \sin(j\pi x/3)dx = -2\pi \sinh(3) \cdot \frac{(-1)^j j}{\pi^2 j^2 + 9}$.

$$e^x = \frac{\sinh 3}{3} + 6\sinh 3 \sum_{j=1}^{\infty}\frac{(-1)^j}{\pi^2 j^2 + 9}\cos(\frac{j\pi x}{3})$$
$$- 2\pi \sinh 3 \sum_{j=1}^{\infty}\frac{(-1)^j j}{\pi^2 j^2 + 9}\sin(\frac{j\pi x}{3}),\ -3 < x < 3$$

(d) $L = 1$. The function is even. $a_0 = 2\int_0^1 x^2 dx = 2/3$; $a_j = 2\int_0^1 x^2 \cos(j\pi x)dx = \frac{4}{\pi^2} \cdot \frac{(-1)^j}{j^2}$.

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{j=1}^{\infty}(-1)^j\frac{\cos(j\pi x)}{j^2}, -1 \le x \le 1.$$

(e) $L = \pi/3$. THe function is even. $a_0 = \frac{6}{\pi}\int_0^{\pi/3}\cos 2x dx = \frac{3\sqrt{3}}{2\pi}$; $a_j = \frac{6}{\pi}\int_0^{\pi/3}\cos(2x)\cos(3jx)dx = \frac{6\sqrt{3}}{\pi} \cdot \frac{(-1)^{j+1}}{9j^2 - 4}$.

$$\cos 2x = \frac{3\sqrt{3}}{4\pi} + \frac{6\sqrt{3}}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{9j^2 - 4}\cos(3jx),\ -\frac{\pi}{3} \le x \le \frac{\pi}{3}.$$

(f) $L = 1$. $a_0 = \int_{-1}^{1} \sin(2x - \pi/3)dx = -\frac{\sqrt{3}}{2}\sin 2$; $a_j = \int_{-1}^{1} \sin(2x - \pi/3)\cos(j\pi x)dx = 2\sqrt{3}\sin 2 \cdot \frac{(-1)^j}{\pi^2 j^2 - 4}$; $b_j = \int_{-1}^{1} \sin(2x - \pi/3)\sin(j\pi x)dx = -\pi \sin 2 \cdot \frac{(-1)^j j}{\pi^2 j^2 - 4}$.

$$\sin(2x - \pi/3) = -\frac{\sqrt{3}}{4}\sin 2 + 2\sqrt{3}\sin 2 \sum_{j=1}^{\infty} \frac{(-1)^j}{\pi^2 j^2 - 4}\sin(j\pi x),\ -1 < x < 1.$$

3. Find the Fourier series.

(a) The function is even with average value 1/2 so $a_0 = 1$ and $a_j = 2\int_0^1 (1 - x)\cos(j\pi x)dx = \frac{2}{\pi^2} \cdot \frac{1 + (-1)^{j+1}}{j^2}$.

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{j=1}^{\infty} \frac{\cos(2j-1)\pi x}{(2j-1)^2},\ -1 \leq x \leq 1.$$

(b) The function is even with average value 1 so $a_0 = 2$ and $a_j = 2\int_0^2 x\cos(j\pi x/2)dx = \frac{4}{\pi^2} \cdot \frac{(-1)^j - 1}{j^2}$.

$$|x| = 1 - \frac{8}{\pi^2}\sum_{j=1}^{\infty} \frac{\cos((2j-1)\pi x/2)}{(2j-1)^2},\ -2 \leq x \leq 2.$$

5. $L = 1$. $a_0 = 2\int_1^1 x^2 - x + \frac{1}{6}dx = 0$. $a_j = 2\int_0^1 (x^2 - x + \frac{1}{6})\cos j\pi x dx = \frac{2}{\pi^2} \cdot \frac{1 + (-1)^j}{j^2}$. Therefore, $x^2 - x + \frac{1}{6} = \frac{1}{\pi^2}\sum_{j=1}^{\infty} \frac{\cos 2j\pi x}{j^2}$, $0 \leq x \leq 1$.

7. Since f has period 2, the Fourier series is $f(x) = \cos \pi x$.

5.5 Orthogonal Functions

1. Verify the functions are orthogonal on the given interval.

(a) $\int_{-\pi}^{\pi} \sin 2x \cos 3x dx = \left[\frac{1}{2}\cos x - \frac{1}{10}\cos 5\right]_{-\pi}^{\pi} = 0.$

(b) $\int_0^{\pi} \sin 2x \sin 4x dx = \left[\frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x\right]_0^{\pi} = 0.$

(c) $\int_{-1}^{1} x^2 \cdot x^3 dx = [x^6/6]_{-1}^{1} = 0.$

(d) $\int_{-2}^{2} x\cos 2x dx = \left[\frac{1}{2}\cos 2x + \frac{1}{2}x\sin 2x\right]_{-2}^{2} = 0.$

3. Observe that

$$\begin{aligned}\|f-g\|^2 &= \langle f-g, f-g\rangle \\ &= \langle f, f\rangle - 2\langle f, g\rangle + \langle g, g\rangle \\ &= \|f\|^2 - 2\langle f, g\rangle + \|g\|^2\end{aligned}$$

It follows that $\|f-g\|^2 = \|f\|^2 + \|g\|^2$ if and only if $\langle f, g\rangle = 0$.

5. A calculation like the one in Exercise 3 will verify that

$$\phi(\lambda) = \|f + \lambda g\|^2 = \|f\|^2 + 2\lambda\langle f, g\rangle + \lambda^2\|g\|^2.$$

Since $\phi'(\lambda) = 2\langle f, g\rangle + 2\lambda\|g\|^2$, the function ϕ is minimized when λ has the value $\lambda_0 = -\frac{\langle f,g\rangle}{\|g\|^2}$. Because $\phi(\lambda) \geq 0$ for all λ, $\phi(\lambda_0) \geq 0$ as well, so

$$\|f\|^2 - 2\cdot\frac{\langle f, g\rangle}{\|g\|^2}\cdot\langle f, g\rangle + \left(\frac{\langle f, g\rangle}{\|g\|^2}\right)^2\|g\|^2 \geq 0.$$

This simplifies to $\|f\|^2\|g\|^2 \geq \langle f, g\rangle^2$ which, in turn, implies the desired inequality: $\|f\|\|g\| \geq |\langle f, g\rangle|$.

Chapter 6

Sturm-Liouville Problems and Boundary Value Problems

6.1 What is a Sturm-Liouville Problem?

1. (a)

$$(1 - x^2)\mu'' - 2x\mu' + p(p-1)\mu = 0\,.$$

(b)

$$\mu'' + 2x\mu' + 4\mu = 0\,.$$

(d)

$$x\mu'' + (x+1)\mu' + (1+p)\mu = 0\,.$$

3. We note that the adjoint equation is

$$\mu'' + (2x + 3/x)\mu' + (-2 - 3/x^2)\mu = 0\,. \qquad (*)$$

Guessing that the equation has a solution of the form $\mu(x) = x^a$, we find the solution $\mu(x) = x$. So we multiply $(*)$ by x to obtain

$$xy'' - (2x^2 + 3)y' - 4xy = 0\,.$$

Fortuitously, this can be written as

$$\frac{d}{dx}\big(xy' - (2x^2 + 4)y\big) = 0$$

(this is the concept of exactness of a second order equation). Thus

$$xy' - (2x^2 + 4)y = C\,.$$

If we conveniently take $C = 0$, then we can solve by separation of variables to obtain

$$y = Cx^4 e^{x^2}\,.$$

5. **(a)** If the equation is self-adjoint then

$$P(x) = P(x) \qquad \text{and} \qquad Q(x) = 2P'(x) - Q(x)$$

$$\text{and} \qquad R(x) = P''(x) - Q'(x) + R(x)\,.$$

The second and the third of these equalities both lead to $P'(x) = Q(x)$.

(b) Only Legendre's equation is self-adjoint.

6.2 Analyzing a Sturm-Liouville Problem

1. Multiply the differential equation through by $p(x)$ to obtain

$$p(x)\frac{d^2y}{dx^2} + p(x)\alpha(x)\frac{dy}{dx} + [\lambda\beta(x)p(x) - p(x)\gamma(x)]y = 0\,.$$

This can be rewritten as

$$\frac{dy}{dx}\left(p(x)\frac{dy}{dx}\right) + [\lambda\beta(x)p(x) - p(x)\gamma(x)]y = 0\,.$$

We see that the equation is now in the form of a Sturm-Liouville equation with

$$q(x) = \beta(x)p(x) \qquad \text{and} \qquad r(x) = -p(x)\gamma(x)\,.$$

3. For $\lambda > 0$, the solution of the differential equation has the form

$$y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\,.$$

The two endpoint conditions yield

$$A = 0$$

and

$$A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi - A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0\,.$$

Substituting the first of these into the second yields

$$B\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$$

or

$$\tan\sqrt{\lambda}\pi = -\sqrt{\lambda}\,.$$

A glance at the graphs of

$$\lambda \mapsto \tan\sqrt{\lambda}\pi$$

and

$$\lambda \mapsto -\sqrt{\lambda}$$

reveals that these curves cross infinitely many times. So there are infinitely many eigenvalues. And the eigenfunctions are $\varphi_\lambda(x) = \sin\sqrt{\lambda}x$.

5. (a) Certainly

$$y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\,.$$

Hence

$$A = y(0) = 0$$

and

$$B\sin\sqrt{\lambda}\pi = y(\pi) = 0\,.$$

Thus

$$\sin\sqrt{\lambda}\pi = 0$$

so

$$\lambda = k^2 \quad , \ k = 0, 1, 2, \ldots\,.$$

(b) Surely

$$y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\,.$$

Therefore

$$B\sqrt{\lambda}\cos\sqrt{\lambda}\cdot 0 = y'(0) = 0$$

and

$$B\sin\sqrt{\lambda}\pi + A\cos\sqrt{\lambda}\pi = y(\pi) = 0\,.$$

Thus $\cos\sqrt{\lambda}\pi = 0$ so that $\lambda = (k + 1/2)^2$ for $k = 0, 1, 2, \ldots$.

6.3 Applications of the Sturm-Liouville Theory

1. We calculate that

$$ku_{xx} = \sum_{j=1}^{\infty} -c_j k \frac{\beta_j^2}{L^2} \exp\left(\frac{-\beta_j^2 kt}{L^2}\right) \cos \frac{\beta_j x}{L}$$

and

$$u_t = \sum_{j=1}^{\infty} -c_j \frac{\beta_j^2 k}{L^2} \exp\left(\frac{-\beta_j^2 kt}{L^2}\right) \cos \frac{\beta_j x}{L} .$$

So we see that the equation $u_t = ku_{xx}$ is satisfied. Furthermore,

$$u_x(0,t) = 0$$

trivially. And

$$u_x(L,t) = \sum_{j=1}^{\infty} -\frac{\beta_j}{L} c_j \exp\left(\frac{-\beta)j^2 kt}{L^2}\right) \sin \beta_j$$

as well as

$$u(L,t) = \sum_{j=1}^{\infty} c_j \exp\left(\frac{-\beta_j^2 kt}{L^2}\right) \cos \beta_j .$$

So the equation $hu(L,t) + u_x(L,t) = 0$ leads to

$$hc_j \cos \beta_j = \frac{\beta_j}{L} c_j \sin \beta_j .$$

This is equivalent to

$$\tan \beta_j = \frac{hL}{\beta_j} ,$$

as required.

The c_j may be solved for using the usual orthogonality properties of cosine.

3. Just integrating, we find that

$$u(x,t) = Ax + By + Cxy + D .$$

The condition that u be bounded as $y \to +\infty$ forces $B = C = 0$. So $u(x,t) = Ax + D$. We can solve the condition $u(x,0) = f(x)$ provided f is linear. The other endpoint conditions are

$$D = 0$$

and

$$h(AL + D) + A = 0\,.$$

This last translates to

$$A(hL + 1) = 0\,.$$

If $A = 0$ then the problem is quite trivial. So $h = -1/L$.

7. We multiply the equation through by $p(x)$, so that

$$p(x)\frac{d^2y}{dx^2} + p(x)A(x)\frac{dy}{dx} + [\lambda p(x)B(x) - C(x)p(x)]y = 0\,.$$

Now this can be written as

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + [\lambda p(x)B(x) - p(x)C(x)]y = 0\,.$$

This is in Sturm-Liouville form with $q(x) = p(x)B(x)$ and $r(x) = -p(x)C(x)$.

6.4 Singular Sturm-Liouville

1. Take $\mu = 1$ and $f(x) = x$. We guess a solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^j\,.$$

Plugging this guess into the equation gives

$$-\left(x\sum_{j=1}^{\infty} j a_j x^{j-1}\right)' = x\sum_{j=0}^{\infty} a_j x^j + x\,.$$

Hence

$$-\sum_{j=1}^{\infty} j a_j x^{j-1} - x\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} = \sum_{j=0}^{\infty} a_j x^{j+1} + x\,.$$

We can rewrite this as

$$-\sum_{j=0}^{\infty}(j+1)a_{j+1}x^j - \sum_{j=1}^{\infty}(j+1)ja_{j+1}x^j = \sum_{j=1}^{\infty} a_{j-1}x^j + x\,.$$

Thus we see that

$$\sum_{j=1}^{\infty}\left[-(j+1)^2 a_{j+1} - a_{j-1}\right]x^j = x - a_1x^0\,.$$

From this we infer that

$$a_1 = 0$$

and

$$a_2 = \frac{1+a_0}{-4}\,.$$

We have the recursion

$$a_{j+1} = \frac{-1}{(j+1)^2}a_{j-1}\,.$$

So we may calculate that

$$\begin{aligned} a_3 &= 0\,, \\ a_4 &= \frac{1+a_0}{64}\,, \\ a_5 &= 0\,, \\ a_6 &= \frac{-1-a_0}{2304}\,, \end{aligned}$$

etcetera.

3. (a) Using the formula for P_n, we may calculate that

$$\begin{aligned} \int_0^1 \phi_j(x)\phi_k(x)\,dx &= \int_0^1 P_{2j-1}(x)P_{2k-1}(x)\,dx \\ &= \int_0^1 \left(\frac{1}{2^{2j-1}(2j-1)!}\frac{d^{2j-1}}{dx^{2j-1}}\left[(x^2-1)^{2j-1}\right]\right) \\ &\quad \cdot \left(\frac{1}{2^{2k-1}(2k-1)!}\frac{d^{2k-1}}{dx^{2k-1}}\left[(x^2-1)^{2k-1}\right]\right)\,dx\,. \end{aligned}$$

Let us suppose that $j > k$. Then we may integrate by parts $2k$ times, each time taking a derivative off the first parenthetical expression in the integrand and throwing it onto the second parenthetical expression in the integrand. The result is

$$\int_0^1 \left(\frac{1}{2^{2j-1}(2j-1)!} \frac{d^{2j-2k-1}}{dx^{2j+2k-2}} \left[(x^2-1)^{2j-1} \right] \right)$$

$$\cdot \left(\frac{1}{2^{2k-1}(2k-1)!} \frac{d^{2k+2k-1}}{dx^{2j+2k-2}} \left[(x^2-1)^{2k-1} \right] \right) \, dx \, .$$

Note that, because of the factors $(1-x^2)^\lambda$, the boundary terms in the integration by parts vanish. Also observe that $4k-1$ (the number of derivatives in the second term) exceeds $4k-2$ (the power of x in the second term. So in fact the parenthetical expression on the right now vanishes. In conclusion, the inner product of ϕ_j and ϕ_k is 0.

(b) $y = \sum_{j=1}^{\infty} \frac{c_j}{\lambda_j - \mu} P_{2j-1}(x)$, where the c_j are calculated using the orthogonality of the P_{2j-1}.

5. The series solutions in Exercise 3b) are not bounded.

7. One may use power series methods to solve the equation

$$xu'' + (1-x)u' + \lambda u = 0$$

to see that the solution is always an infinite series, never a polynomial.

Chapter 7

Partial Differential Equations and Boundary Value Problems

7.1 Introduction and Historical Remarks

7.2 Eigenvalues and the Vibrating String

1. Find the eigenvalues and eigenfunctions for $y'' + \lambda y = 0$. In all of the problems $\lambda > 0$ and the general solution is $y = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$.

(a) $y(0) = 0$ implies that $B = 0$ so $y = A\sin\sqrt{\lambda}x$. The condition $y(2\pi) = 0$ implies that $\sqrt{\lambda}\cdot 2\pi = n\pi$ for some integer n and the eigenvalues are $\lambda_n = n^2/4, n = 1, 2, 3, \cdots$. The eigenfunctions are $y_n = \sin(nx/2)$.

(b) $y(0) = 0$ implies that $B = 0$ so $y = A\sin\sqrt{\lambda}x$. The condition $y(2\pi) = 0$ implies that $\sqrt{\lambda}\cdot 2\pi = n\pi$ for some integer n and the eigenvalues are $\lambda_n = n^2/4, n = 1, 2, 3, \cdots$. The eigenfunctions are $y_n = \sin(nx/2)$.

(c) $y(0) = 0$ implies that $B = 0$ so $y = A\sin\sqrt{\lambda}x$. The condition $y(1) = 0$ implies that $\sqrt{\lambda} = n\pi$ for some integer n and the eigenvalues are $\lambda_n = n^2\pi^2, n = 1, 2, 3, \cdots$. The eigenfunctions are $y_n = \sin n\pi x$.

(d) $y(0) = 0$ implies that $B = 0$ so $y = A\sin\sqrt{\lambda}x$. The condition $y(L) = 0$ implies that $\sqrt{\lambda}\cdot L = n\pi$ for some integer n and the eigenvalues are $\lambda_n = n^2\pi^2/L^2, n = 1, 2, 3, \cdots$. The eigenfunctions are $y_n = \sin(n\pi x/L)$.

(e) The condition $y(-L) = 0$ and $y(L) = 0$ imply that λ must solve the following two equations.

$$-A\sin\sqrt{\lambda}L + B\cos\sqrt{\lambda}L = 0$$
$$A\sin\sqrt{\lambda}L + B\cos\sqrt{\lambda}L = 0$$

It follows that either A or B must be 0 (verify). If $B = 0$, then $\sqrt{\lambda}L = n\pi$ and there are eigenvalues $n^2\pi^2/L^2$ with the eigenfunctions $\sin(n\pi x/L), n = 1, 2, 3, \cdots$. If $A = 0$, then $\sqrt{\lambda}L = (2n-1)\pi/2$ and there are eigenvalues $(2n-1)^2\pi^2/4L^2$ with the eigenfunctions $\cos((2n-1)\pi x/2L), n = 1, 2, 3, \cdots$.

(f) Make the substitution $t = x - a$ and, according to part (d), the eigenvalues are $n^2\pi^2/(b-a)^2$ with eigenfunctions $y_n(t) = \sin n\pi t/(b-a)$. In terms of the variable x the eigenvalues are the same the the eigenfunctions are $y_n(x) = \sin n\pi(x-a)/(b-a)$.

The solution technique used in (f) can also be applied to the problem in part (e) yielding eigenvalues $\lambda_n = n^2\pi^2/4L^2$ and eigenfunctions $y_n = \sin n\pi(x+L)/2L$. Show that the solution formulas are equivalent to the ones found above.

3. The solution has the form $y(x,t) = F(x+at) + G(x-at)$.

(a) The condition $y(x,0) = f(x)$ implies that $F(x) + G(x) = f(x)$. The condition $y_t(x,0) = 0$ implies that $aF'(x) - aG'(x) = 0$. Thus $F'(x) = G'(x)$ so there is a constant C such that $F(x) - G(x) = C$. It follows that $2F(x) = f(x) + C$ and $2G(x) = f(x) - C$. This permits us to express the solution as

$$\begin{aligned} y(x,t) &= F(x+at) + G(x-at) \\ &= \frac{1}{2}[f(x+at) + C] + \frac{1}{2}[f(x-at) - C] \\ &= \frac{1}{2}[f(x+at) + f(x-at)] \end{aligned}$$

(b) If $y(0,t) = 0$ for all t, then $f(at) = -f(-at)$ for all t and f is an odd function. If, in addition, $y(\pi, t) = 0$ for all t, then

$f(\pi+at) = -f(\pi-at) = f(at-\pi)$ for all t. Given x choose t such that $x = at-\pi$. Then $f(x) = f(at-\pi) = f(at+\pi) = f(x+2\pi)$.

(c) Since f is odd, $f(0) = f(-0) = -f(0)$. Thus $2f(0) = 0$ implying that $f(0) = 0$. Because f has period 2π, $f(-\pi) = f(\pi)$. It is also true that $f(-\pi) = -f(\pi)$ (f is odd). Consequently, $f(\pi) = 0$.

(d) Bernoulli's solution is $y(x,t) = \sum_{j=1}^{\infty} b_+j \sin jx \cos jt$. Using the given identity this can be expressed in the form $y(x,t) = \sum_{j=1}^{\infty} b_j \cdot \frac{1}{2}[\sin j(x+at) + \sin j(x-at)] = \frac{1}{2}[f(x+at) + f(x-at)]$, where $f(x) = \sum_{j=1}^{\infty} b_j \sin jx$.

5. If the initial shape is $f(x) = c\sin x$, then the solution is $y(x,t) = c\sin x \cos t$. Thus for every t the shape of the solution is a single arch of a sine curve. Similarly, if the initial shape is $f(x) = c\sin nx$, then the solution is $y(x,t) = c\sin nx \cos nt$. Thus for every t the shape of the solution is a sine curve with nodes at $x_k = \pi/k, k = 1, 2, \cdots, n-1$.

7.3 The Heat Equation

1. The Fourier series solution to $a^2 w_{xx}(x,t) = w_t(x,t)$ satisfying the boundary conditions $w(0,t) = w(\pi,t) = 0$ is $W(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2a^2t} \sin jx$. This can be seen by substituting $w(x,t) = u(x)v(t)$ and separating variables. An easier way is to make the change of variables $\tau = a^2 t$ in the heat equation to obtain $w_{xx}(x,\tau) = w_\tau(x,\tau)$ having the solution $W(x,\tau) = \sum_{j=1}^{\infty} b_j e^{-j^2\tau} \sin jx$ obtained in the text, and then express the solution in terms of t.

Now let $w(x,t) = W(x,t) + g(x)$, where $g(x) = w_1 + \frac{1}{\pi}(w_2 - w_1)x$. By the superposition principal, w is also a solution to the heat equation and, since $W(0,t) = W(\pi,t) = 0$, $w(x,t)$ satisfies the boundary conditions $w(0,t) = g(0) = w_1, w(\pi,t) = g(\pi) = w_2$.

The initial temperature distribution, $w(x,0) = f(x)$, determines the values of the coefficients b_j as follows. Since $f(x) = \sum_{j=1}^{\infty} b_j \sin jx + g(x)$ the coefficients must be chosen so that $\sum_{j=1}^{\infty} b_j \sin jx = f(x) - g(x)$. Consequently, $b_j = \frac{2}{\pi}\int_0^{\pi}(f(x) - g(x)) \sin jx dx$, and the solution is $w(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2a^2t} \sin jx + g(x)$.

3. First find separated solutions to $a^2 w_{xx} = w_t + cw$. The process is made easier by rescaling the variables. Let $\tau = ct$ and $z = \frac{\sqrt{c}}{a}x$ to obtain the equation $w_{zz} = w_\tau + w$.

Substitute $w(z,\tau) = \alpha(z)\beta(\tau)$ to get $\alpha''\beta = \alpha\beta' + \alpha\beta$. Divide by $\alpha\beta$ and the variables are separated: $\frac{\alpha''}{\alpha} = \frac{\beta'}{\beta} + 1$. Since the left side depends only on z and the right side depends only on τ, there is a constant K such that $\frac{\alpha''}{\alpha} = K = \frac{\beta'}{\beta} + 1$. Thus $\alpha'' = K\alpha$ and $\beta' = (K-1)\beta$. This implies that $\beta(\tau) = Ce^{(K-1)\tau}$. Regarding α, the boundary conditions $w(0,t) = w(\frac{\sqrt{c}}{a}\pi, t) = 0$ require that $\alpha(0) = 0$ and $\alpha(\frac{\sqrt{c}}{a}\pi) = 0$. This forces $K = -a^2n^2/c$ for some integer n and $\alpha(z) = A\sin(anz/\sqrt{c})$. In terms of z and τ the separated solutions are $w(z,r) = e^{-(a^2n^2/c+1)\tau}\sin(anz/\sqrt{c})$. In terms of x and t, $w(x,t) = e^{-ct}e^{-n^2a^2t}\sin nx$. Taking linear combinations we have the formal series solution $w(x,t) = e^{-ct}\sum_{j=1}^{\infty} b_j e^{-j^2a^2t}\sin jx$.

The initial condition $w(x,0) = f(x)$ becomes $\sum_{j=1}^{\infty} b_j \sin jx = f(x)$ which is satisfied provided $b_j = \frac{2}{\pi}\int_0^\pi f(x)\sin jx dx$.

5. We seek separated solutions to the heat equation: $a^2 w_{xx} = w_t$, satisfying the boundary conditions $w_x(0,t) = 0 = w_x(\pi,t)$. Substitute $w(x,t) = \alpha(x)\beta(t)$ to get $a^2\alpha''\beta = \alpha\beta'$ or $\frac{\alpha''}{\alpha} = \frac{\beta'}{\alpha^2\beta}$. THus there is a constant K such that $\frac{\alpha''}{\alpha} = K = \frac{\beta'}{a^2\beta}$. That is, $\alpha'' = K\alpha$ and $\beta' = Ka^2\beta$, so $\beta(t) = Ce^{Ka^2t}$. Since the temperature is not expected to grow exponentially with time we assume $K \le 0$ so $\alpha(x) = \sin\sqrt{-K}x$ or $\alpha(x) = \cos\sqrt{-K}x$ or $\alpha(x) = C$, a constant. The last possibility corresponds to $K = 0$.

The boundary conditions require $\alpha'(0) = 0 = \alpha'(\pi)$. Consequently $\alpha(x) = C$, a constant, or $\alpha(x) = \cos\sqrt{-K}x$ with K chosen so that $\alpha'(0) = -\sqrt{K}\sin\sqrt{-K}\pi = 0$. Therefore, the eigenvalues are $K = -n^2, n = 0, 1, 2, \cdots$. The separated solutions are $w(x,t) = e^{-n^2a^2t}\cos nx$. Therefore, the series solution is $w(x,t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j e^{-j^2a^2t}\cos jx$ where the coefficients a_j satisfy $w(x,0) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos jx = f(x)$. That is, $a_j = \frac{2}{\pi}\int_0^\pi f(x)\cos jx dx$.

7. Let (x,y,z) be the point at the center of the box R. Its six faces are centered at $x \pm \Delta x/2, y, z), (x, y \pm \Delta y/2, z)$, and $(x, y, z \pm \Delta/2)$ respectively. The temperature at the center of the box at time t is $w(x,y,z,t)$. Let Δw denote the change in temperature at the center corresponding to a small change in time Δt. The corresponding change in total heat energy in R is approximately proportional to $\Delta w \cdot \Delta V$ where $\Delta V = \Delta x \Delta y \Delta z$. By the principal of conservation of energy this change must equal the heat energy that has flowed into the box across its six faces. Taking into account the fact that the heat flow across a face is proportional to the area of the face, the temperature gradient across the face, and the length of the time interval, we have

$$\begin{aligned}\Delta w \cdot \Delta V \approx a^2[&\partial_x w(x+\Delta x/2,y,z,t) - \partial_x w(x-\Delta x/2,y,z,t))\Delta y \Delta z \\ &+ (\partial_y w(x,y+\Delta y/2,z,t) - \partial_y w(x,y-\Delta y/2,t))\Delta x \Delta z \\ &+ (\partial_z w(x,y,z+\Delta z/2,t) - \partial_z w(x,y,z-\Delta z/2,t))\Delta x \Delta y]\Delta t\end{aligned}$$

Divide both sides by $\Delta V \cdot \Delta t$ to obtain

$$\begin{aligned}\frac{\Delta w}{\Delta t} \approx a^2[&\frac{\partial_x w(x+\Delta x/2,y,z,t) - \partial_x w(x-\Delta x/2,y,z,t)}{\Delta x} \\ &+ \frac{\partial_y w(x,y+\Delta y/2,z,t) - \partial_y w(x,y-\Delta y/2,z,t)}{\Delta y} \\ &+ \frac{\partial_z w(x,y,z+\Delta/2,t) - \partial_z w(x,y,z-\Delta z/2,t)}{\Delta z}]\end{aligned}$$

Now let $\Delta t \to 0$, then shrink the box to its center to get

$$\frac{\partial w}{\partial t} = a^2 \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right].$$

7.4 The Dirichlet Problem for a Disk

1. Solve the Dirichlet problem for the unit disk for the given boundary function $f(\theta)$.

(a) $f(\theta) = \cos\theta/2$ is even; $a_j = \frac{2}{\pi}\int_0^\pi \cos\theta/2 \cdot \cos j\theta d\theta = \frac{4}{\pi} \cdot \frac{(-1)^{j+1}}{4j^2-1}$, $a_0 = 4/\pi$. Therefore,

$$w(r,\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^j}{4j^2-1} r^j \cos j\theta.$$

(b) $f(\theta) = \theta$ is odd; $b_j = \frac{2}{\pi}\int_0^\pi \theta \sin j\theta d\theta = 2 \cdot \frac{(-1)^{j+1}}{j}$. Therefore,

$$w(r, \theta) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} r^j \sin j\theta.$$

(c) $f(\theta)$ is neither even nor odd; $a_j = \frac{1}{\pi}\int_0^\pi \sin\theta \cos j\theta d\theta = -\frac{1}{\pi} \cdot \frac{1+(-1)^j}{j^2-1}$, $j \neq 1; a_0 = 2/\pi, a_1 = 0; b_j = \frac{1}{\pi}\int_0^\pi \sin\theta \sin j\theta d\theta = 0, j \neq 1; b_1 = 1/2$. Therefore,

$$w(r, \theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} r^{2j} \cos 2j\theta + \frac{1}{2} r \sin\theta.$$

(d) The function $f(\theta) - 1/2$ is odd; $a_0/2 = 1/2; a_j = 0, j > 1; b_j = \frac{1}{\pi}\int_0^\pi \sin j\theta d\theta = \frac{1}{\pi} \cdot \frac{(-1)^{j+1}+1}{j}$. Therefore,

$$w(r, \theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{1}{2j - 1} r^{2j-1} \sin(2j - 1)\theta.$$

(e) $f(\theta) = \theta^2/4$ is even; $a_j = \frac{2}{\pi}\int_0^\pi \theta^2/4 \cos j\theta d\theta = \frac{(-1)^j}{j^2}, a_0 = \pi^2/6$. Therefore,

$$w(r, \theta) = \frac{\pi^2}{12} + \sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} r^j \cos j\theta.$$

3. Let the circle C be centered at (x_0, y_0) (Cartesian coordinates) with radius R. The function $u(x, y) = w(x_0 + x, y_0 + y)$ is harmonic on the disk centered at the origin of radius R. According to the Poisson integral formula for this disk (Exercise 2), u's value at the center of the disk: $(0, \theta)$, (polar coordinates) is given by $u(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R, \phi) d\phi$. In terms of the original function w this formula can be expressed in the following form.

$$w(x_0, y_0) = \frac{1}{2\pi R}\int_{-pi} \pi w(x_0 + R\cos\phi, y_0 + R\sin\phi) R d\phi.$$

5. Neither Maple nor Mathematica can obtain useful formulas for the Poisson integrals.

7.5 Sturm-Liouville Problems

1. The differential equation $(\mu P y')' + (Sy)' = 0$ is equivalent to the equation $(\mu P)y'' + ((\mu P)' + S)y' + S'y = 0$. Therefore, the functions μ and S must be chosen so that the following two equations are satisfied: (1) $(\mu P)' + S = \mu Q$ and (2) $S' = \mu R$. Differentiate the first equation and use the second equation to see that μ must solve $(\mu P)'' + \mu R = (\mu Q)'$. This is the *adjoint equation* in disguise.

(a) The adjoint to Legendre's equation is the same equation. Legendre's equation is *self-adjoint*.

(b) The adjoint for Bessel's equation is $x^2\mu'' + 3x\mu' + (x^2 + 1 - p^2)\mu = 0$.

(c) Hermite's equation has adjoint $\mu'' + 2x\mu' + (p + 1)\mu = 0$.

(d) The adjoint to Laguerre's equation is $x\mu'' + (1 + x)\mu' + (p + 1)\mu = 0$.

3. This is the same question as Exercise 1. Parts (a), (b), (d) and (f) are done above.

(c) Chebyshev's equation has the adjoint $(1 - x^2)\mu'' - 3x\mu' + (p^2 - 1)\mu = 0$.

(e) The adjoint to Airy's equation is the same equation. Airy's equation is self-adjoint.

5. Let $pv'' + qv' + rv = 0$ be the adjoint of the adjoint. Then $p = P, q = 2P' - (2P' - Q) = Q$, and $r = P'' - (2P' - Q)' + P'' - Q' + R = R$. Therefore, the adjoint of the adjoint of $Py'' + Qy' + Ry = 0$ is $Pv'' + Qv' + Rv = 0$.

7. Divide the differential equation by $P(x)$ and suppose that the function μ has the property that $\mu y'' + \mu\frac{Q}{P}y' + \mu\frac{R}{P}y = 0$ can be put into self-adjoint form. That is, $\mu y'' + \mu\frac{Q}{P}y' = (\mu y')'$. This equation simplifies to $\mu' = \mu\frac{Q}{P}$ or $\frac{\mu'}{\mu} = \frac{Q}{P}$. Consequently, $\ln\mu = \int\frac{Q}{P}dx$ or $\mu = e^{\int Q/P dx}$. Therefore, the original equation $Py'' + Qy' + Ry = 0$ can be made self-adjoint by first dividing by P and then multiplying by $e^{\int Q/P dx}$.

Chapter 8

Laplace Transforms

8.1 Introduction

1. Evaluate the integrals in the Laplace transform table. Unless otherwise indicated it is assumed that $p > 0$.

$$L[1] = \int_0^\infty e^{-px} dx = \left[-\frac{e^{-px}}{p} \right]_{x=0}^{x\to\infty} = \frac{1}{p}.$$

$$L[x] = \int_0^\infty x e^{-px} dx = \left[-(\frac{1}{p^2} + \frac{x}{p}) e^{-px} \right]_{x=0}^{x\to\infty} = \frac{1}{p^2}.$$

$$L[x^n] = \frac{n!}{p^{n+1}}. \text{ See the calculation in this section.}$$

$$L[e^{ax}] = \int_0^\infty e^{ax} e^{-px} dx = \left[\frac{e^{-(p-a)x}}{-(p-a)} \right]_{x=0}^{x\to\infty} = \frac{1}{p-a},\ p > a.$$

$$L[\sin ax] = \int_0^\infty e^{-px} \sin ax dx = -\left[\frac{p \sin ax + a \cos ax}{p^2 + a^2} e^{-px} \right]_{x=0}^{x\to\infty} = \frac{a}{p^2 + a^2}.$$

$$L[\cos ax] = \int_0^\infty e^{-px} \cos ax dx = -\left[\frac{p \cos ax - a \sin ax}{p^2 + a^2} e^{-px} \right]_{x=0}^{x\to\infty} = \frac{p}{p^2 + a^2}.$$

$$L[\sinh ax] = \frac{1}{2}\int_0^\infty (e^{ax} - e^{-ax})e^{-px}dx = -\frac{1}{2}\left[\frac{e^{-(p-a)}}{p-1} - \frac{e^{-(p+a)}}{p+a}\right]_{x=0}^{x\to\infty} = \frac{a}{p^2-a^2},\ p > |a|.$$

$$L[\cosh ax] = \frac{1}{2}\int_0^\infty (e^{ax} + e^{-ax})e^{-px}dx = -\frac{1}{2}\left[\frac{e^{-(p-a)}}{p-1} + \frac{e^{-(p+a)}}{p+a}\right]_{x=0}^{x\to\infty} = \frac{a}{p^2-a^2},\ p > |a|.$$

3. Since $\sin^2 ax = \frac{1}{2}(1 - \cos 2ax)$, and the Laplace transform is a linear operator,

$$\begin{aligned} L[\sin^2 ax] &= \frac{1}{2}\left(L[1] - L[\cos 2ax]\right) \\ &= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4a^2}\right) = \frac{2a^2}{p(p^2+4a^2)}. \end{aligned}$$

Similarly, $L[\cos^2 ax] = \frac{1}{2}(L[1] + L[\cos 2ax]) = \frac{2a^2+p^2}{p(p^2+4a^2)}$.

Since $\sin^2 ax + \cos^2 ax = 1$, the two Laplace transforms should add up to $1/p$ (verify).

4. **(a)** $L[10] = 10L[1] = \frac{10}{p}$

(c) $L[2e^{3x} - \sin 5x] = 2L[e^{3x}] - L[\sin 5x] = 2 \cdot \frac{1}{p-3} + \frac{5}{p^2+25}$

(e) $L[x^6 \sin^2 3x + x^6 \cos^2 3x] = L[x^6] = \frac{6!}{p^7}$

5. Find the function whose Laplace transform is given.

(a) Since $L[x^3] = \frac{6}{p^4}$, the function $f(x) = \frac{30}{6}x^3$ transforms to $\frac{30}{p^4}$.

(b) Since $L[e^{-3x}] = \frac{1}{p+3}$, the function $f(x) = 2e^{-3x}$ transforms to $\frac{2}{p+3}$.

(c) Since $L[x^2 + \sin 2x] = \frac{2}{p^3} + \frac{2}{p^2+4}$, the function $f(x) = 2x^2 + 3\sin 2x$ will transform to $\frac{4}{p^3} + \frac{6}{p^2+4}$.

(d) Using the technique of partial fractions, $\frac{1}{p^2+p} = \frac{1}{p} - \frac{1}{p+1}$. Therefore, the function $f(x) = 1 - e^{-x}$ will transform to $\frac{1}{p^2+p}$.

(e) Using partial fractions, $\frac{1}{p^4+p^2} = \frac{1}{p^2} - \frac{1}{p^2+1}$. Therefore, the function $f(x) = x - \sin x$ will transform to $\frac{1}{p^4+p^2}$.

7. Male is unable to calculate any of the transforms; Mathematica can calculate the transforms for part (a) and part (c). The following code shows how Mathematica calculates the Laplace transform for part (c).

LaplaceTransform[Sin[Log[x]],x,p]

$\frac{1}{2}p^{-1-i}(p^{2i}\text{Gamma}[-i] + \text{Gamma}[i])$

8.2 Applications to Differential Equations

1. The Laplace transforms are displayed in the table. The transforms for (d) and (g) were obtained using a property that is introduced in the next section: $L[x^n f(x)] = (-1)^n \frac{d^n}{dp^n} L[f(x)]$.

Table 8.1: Some Laplace transforms

	Function	Transform
(a)	$x5e^{-2x}$	$\frac{120}{(p+2)^6}$
(b)	$(1-x^2)e^{-x}$	$\frac{1}{p+1} - \frac{2}{(p+1)^3}$
(c)	$e^{-x}\sin x$	$\frac{1}{(p+1)^2+1}$
(d)	$x \sin 3x$	$\frac{6p}{(p^2+9)^2}$
(e)	$e^{3x}\cos 2x$	$\frac{p-3}{(p-3)^2+4}$
(f)	xe^x	$\frac{1}{(p-1)^2}$
(g)	$x^2 \cos x$	$\frac{2p(p^2-3)}{(p^2+1)^3}$
(h)	$\sin x \cos x$	$\frac{1}{p^2+4}$

3. Use the Laplace transform to solve the given IVPs. Each solution begins with the transform of the IVP. We let $Y = L[y]$.

(a) $pY - 0 + Y = \frac{1}{p-2}$ implies that $Y = \frac{1}{(p+1)(p+2)}$. Using partial fractions, $Y = \frac{1}{3}\left(\frac{1}{p-2} - \frac{1}{p+1}\right)$. Therefore, the solution is $y =$

$\frac{1}{3}(e^{2x} - e^{-x})$.

(b) $p^2Y - p\cdot 0 - 3 - 4\cdot(pY - 0) + 4Y = 0$ implies that $(p^2 - 4p + 1)Y - 3 = 0$. Therefore, $Y = \frac{3}{p^2-4p+4} = \frac{3}{(p-2)^2}$ and the solution is $y = 3xe^{2x}$.

(c) $p^2Y - p\cdot 0 + 2\cdot(pY - 0) + 2Y = \frac{2}{p}$ implies that $(p^2 + 2p + 2)Y - 1 = \frac{2}{p}$. Therefore, $y = \frac{1+2/p}{p^2+2p+2} = \frac{p+2}{p(p^2+2p+2)}$. Use partial fractions and complete the square to obtain $Y = \frac{1}{p} - \frac{p+1}{p^2+2p+2} = \frac{1}{p} - \frac{p+1}{(p+1)^2+1}$. Therefore, the solution is $y = 1 - e^{-x}\cos x$.

(d) $p^2Y - p\cdot 0 - 1 + pY - 0 = \frac{6}{p^3}$ implies that $(p^2 + p)Y - 1 = \frac{6}{p^3}$. Therefore, $Y = \frac{1+6/p^3}{p^2+p} = \frac{p^3+6}{p^3(p^2+p)}$. Using partial fractions (the correct decomposition is of the form $\frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p^4} + \frac{E}{p+1}$, we obtain the solution $y = -5 + 6x - 3x^2 + x^3 + 5e^{-x}$.

(e) $p^2Y - p\cdot 0 - 3 + 2\cdot(pY - 0) + 5Y = \frac{3}{(p+1)^2+1}$ implies that Y satisfies $(p^2 + 2p + 5)Y - 3 = \frac{3}{p^2+2p+5}$. Therefore, $Y = \frac{3+\frac{3}{p^2+2p+2}}{p^2+2p+2} = \frac{3p^2+6p+9}{(p^2+2p+2)(p^2+2p+5)}$. The partial fraction decomposition is $Y = \frac{1}{p^2+2p+2} + \frac{2}{p^2+2p+5} = \frac{1}{(p+1)^2+1} + \frac{2}{(p+1)^2+4}$. Therefore, $y = e^{-x}(\sin x + \sin 2x)$.

5. Let $y(x) = \int_0^x f(t)dt$ and $F(p) = L[f(x)]$. By the Fundamental theorem, $y'(x) = f(x)$ so $F(p) = L[y'(x)] = pL[y(x)] - y(0) = pL[\int_0^x f(t)dt]$. Consequently, $L[\int_0^x f(t)dt] = \frac{F(p)}{p}$, as desired. For the purpose of finding an inverse transform, this can be expressed as $L^{-1}\left[\frac{1}{p}F(p)\right] = \int_0^x f(t)dt$.

This inverse transform formula can be used to avoid partial fractions. For example, $L^{-1}\left[\frac{1}{p(p+1)}\right] = \int_0^x e^{-t}dt = [-e^{-t}]_0^x = 1 - e^{-x}$.

8.3 Derivatives and Integrals of Laplace Transforms

1. $L[x\cos ax] = -L[-x\cos ax] = -\frac{d}{dp}L[\cos ax] = -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right) = \frac{p^2-a^2}{(p^2+a^2)^2}$. To invert $\frac{1}{(p^2+a^2)^2}$ note that $L[x\cos ax] = \frac{p^2+a^2-2a^2}{(p^2+a^2)^2} = \frac{1}{p^2+a^2} - \frac{2a^2}{(p^2+a^2)^2}$.

Therefore, $L\left[x\cos ax - \frac{1}{a}\sin ax\right] = -\frac{2a^2}{(p^2+a^2)^2}$. It follows that

$$\begin{aligned} L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] &= -\frac{1}{2a^2}\left(x\cos ax - \frac{1}{a}\sin ax\right) \\ &= \frac{1}{2a^3}(\sin ax - ax\cos ax). \end{aligned}$$

3. Start with the Laplace transform. Let $y(0) = A$.

(a) $-\frac{d}{dp}[p^2Y - Ap] - 3\frac{d}{dp}[pY] - (pY - A) + 4\frac{dY}{dp} - 9Y = 0$. This evaluates to the equation $(-p^2 - 3p + 4)Y' - (3p + 12)Y = -2A$ which, in turn, simplifies to $(p-1)Y' + 3Y = 2A/(p+4)$. Consequently, $Y = A \cdot \frac{p^2 - 12p + 50\ln(p+4)}{(p-1)^3} + B \cdot \frac{1}{(p-1)^3}$ and, setting $A = 0$, $y = Bx^2e^x$.

(b) $-\frac{d}{dp}[p^2Y - Ap] - 2\frac{d}{dp}[pY] + 3(pY - A) - \frac{dY}{dp} + 3Y = \frac{3}{p+1}$. This evaluates to the equation $(-p^2 - 2p - 1)Y' + (p+1)Y = 2A + \frac{3}{p+1}$ which, in turn, simplifies to $(p+1)Y' - Y = -\frac{2A}{p+1} - \frac{3}{(p+1)^2}$. Consequently, $Y = \frac{A}{p+1} - \frac{1}{(p+1)^2} + B(p+1)$ and, setting $B = 0$, $y = (A + x)e^{-x}$.

5. (a) Let $p = 0$ in the equation $\frac{1}{\sqrt{p^2+1}} = L[J_0(x)] = \int_0^\infty J_0(x)e^{-px}dx$.

(b) Observe that $\frac{1}{\pi}\int_0^\pi \cos(x\cos t)dt = \frac{1}{\pi}\int_0^\pi \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!}(x\cos t)^{2j}dt$. Move the integral across the sum and proceed as follows. To justify the last step use $\binom{2j}{j} = \frac{(2j)!}{(j!)^2}$.

$$\begin{aligned} \frac{1}{\pi} = \int_0^\pi \cos(x\cos t)dt &= \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!}\left(\frac{1}{\pi}\int_0^\pi \cos^{2j} t dt\right)x^{2j} \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{(2j)!} \cdot \frac{1}{2^{2j}}\binom{2j}{j} \cdot x^{2j} \text{ (see below)} \\ &= J_0(x) \text{ (exercise)} \end{aligned}$$

The verification that $\frac{1}{\pi}\int_0^\pi \cos^{2j} t dt = \frac{1}{2^{2j}}\binom{2j}{j}$ goes like this:

$$\begin{aligned}
\int_0^\pi \cos^{2j} t dt &= \int_0^\pi \left[\frac{1}{2}(e^{it} + e^{-it}\right]^{2j} dt \\
&= \frac{1}{2^{2j}} \int_0^\pi \sum_{k=0}^{2j} \binom{2j}{k} (e^{it})^{2j-k}(e^{-it})^k dt \\
&= \frac{1}{2^{2j}} \sum_{k=0}^{2j} \binom{2j}{k} \int_0^\pi e^{2(j-k)it} dt \\
&= \frac{1}{2^{2j}} \cdot \binom{2j}{j} \cdot \pi .
\end{aligned}$$

7. (a) Start with $F(p) = \int_0^\infty e^{-px} f(x)dx = \sum_{j=0}^\infty \int_{j \cdot a}^{(j+1)\cdot a} e^{-px} f(x)dx$, make the change of variable $x = u + j \cdot a$ in the j^{th} integral, and proceed as follows:

$$\begin{aligned}
F(p) &= \sum_{j=0}^\infty \int_0^a e^{-p(u+ja)} f(u + ja)du \\
&= \sum_{j=0}^\infty (e^{-ap})^j \int_0^a e^{-pu} f(u)du \\
&= \int_0^a e^{-pu} f(u)du \cdot \sum_{j=0}^\infty (e^{-ap})^j \\
&= \frac{1}{1 - e^{-ap}} \int_0^a e^{-px} f(x)dx
\end{aligned}$$

(b) The function f has period $a = 2$ so $F(p) = \frac{1}{1-e^{-2p}} \int_0^2 e^{-px} f(x)dx = \frac{1}{1-e^{-2p}} \int_0^1 e^{-px} dx = \frac{1}{1-e^{-2p}} \cdot \left[\frac{e^{-px}}{-p}\right]_{x=0}^{x=1} = \frac{1}{p} \cdot \frac{1-e^{-p}}{1-e^{-2p}} = \frac{1}{p(1+e^{-p}}$.

8.4 Convolutions

1. Since $L^{-1}\left[\frac{1}{p^2+a^2}\right] = \frac{1}{a}\sin ax$, $L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \frac{1}{a^2}\int_0^x \sin a(x-t)\sin at dt$. Therefore,

$$L^{-1}\left[\frac{1}{p^2+a^2}\right] = \frac{1}{a^2}\left(\sin ax\int_0^x \cos at\sin at dt - \cos ax\int_0^x \sin at\sin at dt\right)$$
$$= \frac{1}{a^2}\left(\sin ax\cdot\frac{1}{2a}\sin^2 ax - \cos ax\cdot\frac{1}{2a}(ax-\frac{1}{2}\sin 2ax)\right).$$

This simplifies to the solution found in Exercise 1 of the last section (verify).

3. Divide both sides of ** by p to obtain $\frac{L[f(y)]}{p} = \sqrt{\frac{2g}{\pi}}\cdot\frac{L[T(y)]}{p^{1/2}}$. Take the inverse transform of both sides, using convolution, to get the equation $\int_0^y f(y)dy = \frac{\sqrt{2g}}{\pi}\int_0^y \frac{T(t)}{\sqrt{y-t}}dt$. Then differentiate with respect to y to obtain $f(y) = \frac{\sqrt{2g}}{\pi}\frac{d}{dy}\frac{T(t)}{\sqrt{y-t}}dt$.

If $T(y) = T_0$, a constant, then $f(y) = \frac{\sqrt{2g}}{\pi}\frac{d}{dy}\int_0^y \frac{T_0}{\sqrt{y-t}}dt = \frac{\sqrt{2g}}{\pi}\frac{d}{dy}(2T_0y^{1/2})$. Therefore, $f(y) = \frac{\sqrt{2g}}{\pi}\cdot\frac{T_0}{\sqrt{y}}$, as in **.

5. Take the Laplace transform of the IVP to obtain $p^2Y + a^2Y = F(p)$ where F is the Laplace transform of f. Therefore $Y = F(p)\cdot\frac{1}{p^2+a^2}$ so, using convolution on the right side, $y(x) = \frac{1}{a}\int_0^x f(t)\sin a(x-t)dt$.

8.5 The Unit Step and Impulse Functions

1. Find the convolution of the following pairs of functions.

(a) $a * \sin at = \int_0^t \sin a\tau d\tau = \left[-\frac{\cos a\tau}{a}\right]_0^t = \frac{1}{a}(1-\cos at)$.

(b) $e^{at} * e^{bt} = \int_0^t e^{a(t-\tau)}e^{b\tau}d\tau = e^{at}\int_0^t e^{(b-a)\tau}d\tau = e^{at}\left[\frac{e^{(b-a)t}}{b-a}\right]_0^t = \frac{e^{bt}-e^{at}}{b-a}$.

(c) $t * e^{at} = \int_0^t e^{a(t-\tau)}\tau d\tau = e^{at}\int_0^t \tau e^{-a\tau}d\tau = \frac{e^{at}-1-at}{a^2}$.

(d) $\sin at * \sin bt = L^{-1}\left[\frac{a}{p^2+a^2}\cdot\frac{b}{p^2+b^2}\right]$. Using partial fractions,

$$\begin{aligned}\sin at * \sin bt &= \frac{1}{a^2-b^2}\cdot L^{-1}\left[\frac{ab}{p^2+b^2}-\frac{ab}{p^2+a^2}\right]\\ &= \frac{a\sin bt - b\sin at}{a^2-b^2}.\end{aligned}$$

3. These problems are solved using the impulse response function $h(t) = L^{-1}\left[\frac{1}{z(p)}\right]$. The solution is $y(t) = h(t) * f(t) = \int_0^t h(t-\tau)f(\tau)d\tau$. The details of the integration are not shown.

(a) $h(t) = L^{-1}\left[\frac{1}{(p+2)(p+3)}\right] = L^{-1}\left[\frac{1}{p+2}-\frac{1}{p+3}\right] = e^{-2t}-e^{-3t}$. Therefore, $y(t) = \int_0^t 5e^{3\tau}\cdot[e^{-2(t-\tau)}-e^{-3(t-\tau)}]d\tau = \frac{1}{6}(5e^{-3t}-6e^{-2t}+e^{3t})$.

(b) $h(t) = L^{-1}\left[\frac{1}{(p-2)(p+3)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{p-2}-\frac{1}{p-3}\right] = \frac{1}{5}(e^{2t}-e^{-3t})$. Therefore, $y(t) = \int_0^t \tau\cdot\frac{1}{t}[e^{2(t-\tau)}-e^{-3(t-\tau)}]d\tau = -\frac{1}{180}(t+30t-9e^{2t}+4e^{-3t})$.

(c) $h(t) = L^{-1}\left[\frac{1}{p(p-1)}\right] = L^{-1}\left[\frac{1}{p-1}-\frac{1}{p}\right] = e^t-1$. Therefore, $y(t) = \int_0^t \tau^2\cdot[e^{t-\tau}-1]d\tau = 2e^t-\frac{1}{3}t^3-t^2-2t-2$.

5. Make the substitution $\sigma = t-\tau$ to obtain $f*g = \int_0^t f(t-\tau)g(\tau)d\tau = \int_t^0 f(\sigma)g(t-\sigma)(-d\sigma) = \int_0^t g(t-\sigma)f(\sigma)d\sigma = g*f$.
Regarding associativity, interchange the order of integration, then make the change of variable $\tau-\sigma=\mu$.

$$\begin{aligned}f*[g*h] &= \int_0^t f(t-\tau)\left(\int_0^\tau g(\tau-\sigma)h(\sigma)d\sigma\right)d\tau\\ &= \int_0^t\left(\int_\sigma^t f(t-\tau)g(\tau-\sigma)d\tau\right)h(\sigma)d\sigma\\ &= \int_0^t\left(\int_0^{t-\sigma} f(t-\sigma-\mu)g(\mu)d\mu\right)h(\sigma)d\sigma\\ &= [f*g]*h.\end{aligned}$$

7. The impulse response function is $h(t) = L^{-1}\left[\frac{1}{Lp+R}\right] = \frac{1}{L}e^{-Rt/L}$ and the output current is

$$I(t) = \frac{1}{L}\int_0^t E(\tau)\cdot e^{-R(t-\tau)/L}d\tau = \frac{1}{L}e^{-Rt/L}\int_0^t E(\tau)\cdot e^{R\tau/L}d\tau.$$

(a) $I(t) = \frac{E_0}{L} e^{-Rt/L} \int_0^t u(\tau) \cdot e^{R\tau/L} d\tau = \frac{E_0}{R}(1 - e^{-Rt/L})$.

(b) $I(t) = \frac{E_0}{L} e^{-Rt/L}$.

(c) $I(t) = \frac{E_0}{L} e^{-Rt/L} \int_0^t \sin(\omega\tau) \cdot e^{R\tau/L} d\tau$. This evaluates to

$$I(t) = \frac{E_0}{R^2 + \omega^2 L^2}[\omega L e^{-Rt/L} + R \sin \omega t - \omega L \cos \omega t].$$

Chapter 9

Systems of First-Order Equations

9.1 Introductory Remarks

1. Replace the differential equations with an equivalent system of first-order equations.

(a) $y'' - xy' - xy = 0$; let $y_0 = y, y_1 = y'$ to obtain

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= xy_0 + xy_1. \end{aligned}$$

(b) $y''' = y'' - x^2(y')^2$; let $y_0 = y, y_1 = y', y_2 = y''$ to obtain

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= y_2 \\ y_2' &= -x^2 y_1^2 + y_2. \end{aligned}$$

(c) $xy'' - x^2y' - x^3y = 0$; let $y_0 = y, y_1 = y'$ to obtain

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= x^2 y_0 + xy_1. \end{aligned}$$

(d) $y^{(4)} - xy''' + x^2y'' - x^3y = 1$; let $y_0 = y, y_1 = y', y_2 = y'', y_3 = y'''$ to obtain

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= x^3y_0 - x^2y_2 + xy_3 + 1. \end{aligned}$$

9.2 Linear Systems

1. The Wronskian for (*) is $x_1(t)x_2'(t) - x_1'(t)x_2(t)$. Since $y_1 = x_1'$ and $y_2 = x_2'$ this is also the Wronskian of the two solutions to (**):

$$W(x) = \det\begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} = \det\begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix}.$$

3. (a) If $x = 2e^{4t}$ and $y = 3e^{4t}$, then $x' = 8e^{4t}$ and $y' = 12e^{4t}$. Since $x + 2y = 8e^{4t}$ and $3x + 2y = 12e^{4t}$, the first pair of functions do form a solution to the given homogeneous system.

Verification for the second pair can be made with a similar calculation.

(b) The two solutions are clearly linearly independent. One pair is not a scalar multiple of the other. This can be formally verified by calculating their Wronskian: $W(t) = \det\begin{pmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{pmatrix} = -5e^{3t} \neq 0.$

(c) The verification is straightforward. Simply substitute and simplify.

Since the two solutions in part (a) are linearly independent there is a general solution of the form

$$\begin{cases} x(t) = 2c_1e^{4t} + c_2e^{-t} + 3t - 2 \\ y(t) = 3c_1e^{4t} - c_2e^{-t} - 2t + 3 \end{cases}.$$

5. Differentiate the first equation: $x'' = x' + y'$, and use the second equation to replace y' with y: $x'' = x' + y$. Now use the first equation to eliminate y: $x'' = x' + x' - x$. This second order equation simplifies to $x'' - 2x' + x = 0$ which, by inspection, has the general solution $x(t) = c_1e^t + c_2te^t$. Use the first equation to find y:

$$\begin{aligned} y(t) &= x'(t) - x(t) \\ &= c_1e^t + c_2(te^t + e^t) - c_1e^t - c_2te^t \\ &= c_2e^t \end{aligned}$$

The general solution is

$$\begin{cases} x(t) = c_1e^t + c_2te^t \\ y(t) = c_2e^t \end{cases}$$

7. Substitute (6) to (4)

$$\begin{cases} \frac{d}{dt}[c_1x_1(t) + c_2x_2(t)] = a_1(t)[c_1x_1(t) + c_2x_2(t)] + b_1(t)[c_1y_1(t) + c_2y_2(t)] \\ \frac{d}{dt}[c_1y_1(t) + c_2y_2(t)] = a_2(t)[c_1x_1(t) + c_2x_2(t)] + b_2(t)[c_1y_1(t) + c_2y_2(t)] \end{cases}$$

Carry out the differentiation on the left sides and rearrange the right sides to obtain the following equivalent system.

$$\begin{cases} c_1x_1'(t) + c_2x_2'(t) = c_1[a_1(t)x_1(t) + b_1(t)y_1(t)] + c_2[a_1(t)x_2(t) + b_1(t)y_2(t)] \\ c_1y_1'(t) + c_2y_2'(t) = c_1[a_2(t)x_1(t) + b_2(t)y_1(t)] + c_2[a_2(t)x_2(t) + b_2(t)y_2(t)] \end{cases}$$

This system is satisfied for all t in $[a, b]$ because of the assumption that the functions are solutions to the homogeneous equation.

9.3 Homogeneous Systems with Constant Coefficients

1. Find the general solution to each system.

(a) The auxiliary equation is $m^{-}1 = 0$ with roots $m_1 = -1$ and $m_2 = 1$. With $m_1 = -1$ the algebraic system is

$$\begin{aligned} -2A + 4B = 0 \\ -2A + 4B = 0 \end{aligned}$$

$A = 2$ and $B = 1$ is a nontrivial solution and

$$\begin{cases} x = 2e^{-t} \\ y = e^{-t} \end{cases}$$

is a nontrivial solution to the system of differential equations. With $m_2 = 1$ a similar calculation yields

$$\begin{cases} x = e^{t} \\ y = e^{t} \end{cases}$$

as a second, independent solution. The general solution is

$$\begin{cases} x = 2c_1e^{-t} + c_2e^{t} \\ y = c_1e^{-t} + c_2e^{t}. \end{cases}$$

(b) The auxiliary equation is $m^2 - 6m + 18 = 0$ with roots $m_1 = 3+3i$ and $m_2 = 3 - 3i$. With $m_1 = 3 + 3i$ the algebraic system is

$$\begin{aligned} (1-3i)A - 2B &= 0 \\ 5A + (-1-3i)B &= 0 \end{aligned}$$

$A^* = 2$ and $B^* = 1-3i$ has a nontrivial complex solution. Observe that $A_1 = 2, A_2 = 0$ and $B_1 = 1, B_2 = -3$. This is what is needed to obtain two real solution. See Equations (16) and (17), Section 10.3. The solutions are

$$\begin{cases} x = e^{3t} \cdot 2\cos 3t \\ y = e^{3t}(\cos 3t + 3\sin 3t) \end{cases}$$

and

$$\begin{cases} x = e^{3t} \cdot 2\sin 3t \\ y = e^{3t}(\sin 3t - 3\cos 3t) \end{cases} .$$

The general solution is

$$\begin{cases} x = 2e^{3t}[c_1\cos 3t + c_2\sin 3t] \\ y = e^{et}[c_1(\cos 3t + 3\sin 3t) + c_2(\sin 3t - 3\cos 3t)]. \end{cases}$$

(c) The auxiliary equation is $m^2-6m+9=0$ with roots $m_1=m_2=3$. With $m_1=3$ the algebraic system is

$$\begin{aligned} 2A+4B&=0\\ -A-2B&=0. \end{aligned}$$

$A=2$ and $B=-1$ is a nontrivial solution and

$$\begin{cases} x=2e^{3t}\\ y=-e^{3t} \end{cases}$$

is a nontrivial solution to the system of differential equations. We seek a second solution of the form

$$\begin{cases} x=(A_1+A_2t)e^{3t}\\ y=(B_1+B_2t)e^{3t}. \end{cases}$$

Substitute these into the system of differential equations to obtain

$$\begin{aligned} (3A_1+A_2+3A_2t)e^{3t}&=5(A_1+A_2t)e^{3t}+4(B_1+B_2t)e^{3t}\\ (3B_1+B_2+3B_2t)e^{3t}&=-(A_1+A_2t)e^{3t}+(B_1+B_2t)e^{3t}. \end{aligned}$$

The exponential terms can be canceled and, since these equations are identities in the variable t, it follows that

$$\begin{aligned} 3A_1+A_2&=5A_1+4B_1, 3A_2=5A_2+4B_2\\ 3B_1+B_2&=-A_1+B_1, 3B_2=-A_2+B_2. \end{aligned}$$

The two equations on the right are actually the same: $A_2=-2B_2$, and can be solved with $A_2=2, B_2=-1$. Substitute these values into the equations on the left and they can be solved with $A_1=1, B_1=0$. This yields a second solution of the form

$$\begin{cases} x=(1+2t)e^{3t}\\ y=-te^{3t}. \end{cases}$$

The general solution is

$$\begin{cases} x=[2c_1+c_2(1+2t)]e^{3t}\\ y=-(c_1+c_2t)e^{3t}. \end{cases}$$

(d) The auxiliary equation is $m^2 + 2m = 0$ with roots $m_1 = 0$ and $m_2 = -2$. With $m_1 = 0$ the algebraic system is

$$\begin{aligned} 4A - 3B &= 0 \\ 8A - 6B &= 0 \end{aligned}$$

$A = 3$ and $B = 4$ is a nontrivial solution and

$$\begin{cases} x = 3 \\ y = 4 \end{cases}$$

is a nontrivial (constant) solution to the system of differential equations. With $m_2 = -2$ the algebraic system is

$$\begin{aligned} 6A - 3B &= 0 \\ 8A - 4B &= 0 \end{aligned}$$

$A = 1$ and $B = 2$ is a nontrivial solution and

$$\begin{cases} x = e^{-2t} \\ y = 2e^{-2t} \end{cases}$$

is a second, independent solution. The general solution is

$$\begin{cases} x = 3c_1 + c_2 e^{-2t} \\ y = 4c_1 + 2c_2 e^{-2t}. \end{cases}$$

(e) This is an uncoupled system. By inspection, the general solution is

$$\begin{cases} x = c_1 e^{2t} \\ y = c_2 e^{3t}. \end{cases}$$

(f) The auxiliary equation is $m^2 + 6m + 9 = 0$ with roots $m_1 = m_2 = -3$. With $m_1 = -3$ the algebraic system is

$$\begin{aligned} -A - B &= 0 \\ A + B &= 0. \end{aligned}$$

$A = 1$ and $B = -1$ is a nontrivial solution and

$$\begin{cases} x = e^{-3t} \\ y = -e^{-3t} \end{cases}$$

is a nontrivial solution to the system of differential equations. We seek a second solution of the form

$$\begin{cases} x = (A_1 + A_2 t)e^{-3t} \\ y = (B_1 + B_2 t)e^{-3t}. \end{cases}$$

Substitute these into the system of differential equations to obtain

$$\begin{aligned} -(3A_1 - A_2 + 3A_2 t)e^{-3t} &= -[4A_1 + B_1 + (4A_2 + B_2)t]e^{-3t} \\ -(3B_1 - B_2 + 3B_2 t)e^{-3t} &= [A_1 - 2B_1 + (A_2 - 2B_2)t]e^{-3t}. \end{aligned}$$

The exponential terms can be canceled and, since these equations are identities in the variable t, it follows that

$$\begin{aligned} 3A_1 - A_2 = 4A_1 + B_1, 3A_2 &= 4A_2 + B_2 \\ -3B_1 + B_2 = A_1 - 2B_1, -3B_2 &= A_2 - 2B_2. \end{aligned}$$

The two equations on the right are actually the same: $A_2 = -B_2$, and can be solved with $A_2 = 1, B_2 = -1$. Substitute these values into the equations on the left and they can be solved with $A_1 = -1, B_1 = 0$. This yields a second solution of the form

$$\begin{cases} x = (-1 + t)e^{-3t} \\ y = -te^{-3t}. \end{cases}$$

The general solution is

$$\begin{cases} x = [c_1 + c_2(-1 + t)]e^{-3t} \\ y = -(c_1 + c_2 t)e^{-3t}. \end{cases}$$

(g) The auxiliary equation is $m^2 - 13m + 30 = 0$ with roots $m_1 = 10$ and $m_2 = 3$. With $m_1 = 10$ the algebraic system is

$$\begin{aligned} -3A + 6B &= 0 \\ 2A - 4B &= 0 \end{aligned}$$

$A = 2$ and $B = 1$ is a nontrivial solution and

$$\begin{cases} x = 2e^{10t} \\ y = e^{10t} \end{cases}$$

is a nontrivial solution to the system of differential equations. With $m_2 = 3$ a similar calculation yields

$$\begin{cases} x = 3e^{3t} \\ y = -2e^{3t} \end{cases}$$

as a second, independent solution. The general solution is

$$\begin{cases} x = 2c_1e^{10t} + 3c_2e^{3t} \\ y = c_1e^{10t} - 2c_2e^{3t}. \end{cases}$$

(h) The auxiliary equation is $m^2 - 6m + 13 = 0$ with roots $m_1 = 3 + 2i$ and $m_2 = 3 - 2i$. With $m_1 = 3 + 2i$ the algebraic system is

$$\begin{aligned} (-2 - 2i)A - 2B &= 0 \\ 4A + (2 - 2i)B &= 0. \end{aligned}$$

$A^* = 1 - i$ and $B^* = -2$ is a nontrivial complex solution. Observe that $A_1 = 1, A_2 = -1$ and $B_1 = -2, B_2 = 0$. This is what is needed to obtain two real solutions. These solutions are

$$\begin{cases} x = e^{3t}(\cos 2t + \sin 2t) \\ y = -2e^{3t} \cos 2t \end{cases}$$

and

$$\begin{cases} x = e^{3t}(\sin 2t - \cos 2t) \\ y = -2e^{3t} \sin 2t. \end{cases}$$

The general solution is

$$\begin{cases} x = e^{3t}[c_1(\cos 2t + \sin 3t) + c_2(\sin 2t - \cos 2t)] \\ y = -2e^{3t}(c_1 \cos 2t + c_2 \sin 2t). \end{cases}$$

3. In the Wronskian calculation all terms cancel except those with repeated sines and cosines.

$$W(t) = \det \begin{pmatrix} e^{at}(A_1 \cos bt - A_2 \sin bt) & e^{at}(A_1 \sin bt + A_2 \cos bt) \\ e^{at}(B_1 \cos bt - B_2 \sin bt) & e^{at}(B_1 \sin bt + B_2 \cos bt) \end{pmatrix}$$
$$= e^{2at}[(A_1 \cos bt - A_2 \sin bt)(B_1 \sin bt + B_2 \cos bt)$$
$$(A_1 \sin bt + A_2 \cos bt)(B_1 \cos bt - B_2 \sin bt)]$$
$$= e^{2at}(A_1B_2 \cos^2 bt - A_2B_1 \sin^2 bt + A_1B_2 \sin^2 bt - A_2B_1 \cos^2 bt)$$
$$= e^{2at}(A_1B_2 - A_2B_1)(\cos^2 bt + \sin^2 bt)$$
$$= e^{2at}(A_1B_2 - A_2B_1).$$

If $A_1B_2 - A_2B_1 = 0$, then there is a constant k such that $B_1^* = kA_1^*$ and the system has the solution

$$\begin{cases} x = A_1^*(e^{(a+ib)t} \\ y = kA_1^*e^{(a+ib)t}. \end{cases}$$

But then $y = kx$, which implies that the auxiliary equation has real roots.

5. (a) Substitute

$$\begin{cases} x = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y = v_1(t)y_1(t) + v_2(t)y_2(t) \end{cases}$$

into the forced system. Because

$$\begin{cases} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{cases}$$

solves the homogeneous equation, most of the terms will cancel leaving the following system of equations

$$\begin{cases} v_1'(t)x_1(t) + v_2'(t)x_2(t) = f_1(t) \\ v_1'(t)y_1(t) + v_2'(t)y_2(t) = f_2(t) \end{cases}$$

The fact that the Wronskian of the homogeneous solutions is not zero guarantees that this system has a solution and, with luck, the solution functions can be integrated to yield a particular solution to the forced equation.

(b) Using the homogeneous solutions from Example 10.3.1, and the given forcing functions, the variation of parameters system has the following form.

$$\begin{cases} v_1' e^{-3t} + v_2' e^{2t} = -5t + 2 \\ -4v_1' e^{-3t} + v_2' e^{2t} = -8t - 8 \end{cases}$$

Subtract the second equation from the first to obtain the equation $5v_1' e^{-3t} = 3t + 10$. Therefore, $v_1' = \frac{1}{5}(3t + 10)e^{3t}$, which integrates to

$$v_1 = \frac{1}{5}(t + 3)e^{3t}.$$

The function v_2 can be found by multiplying the first equation by 4 and adding to the second: $5v_2' e^{2t} = -28t$. This integrates to

$$v_2 = \frac{7}{5}(2t + 1)e^{-2t}.$$

The particular solution is

$$\begin{cases} x_p = \frac{1}{5}(t + 3)e^{3t} \cdot e^{-3t} + \frac{7}{5}(2t + 1)e^{-2t} \cdot e^{2t} \\ y_p = \frac{1}{5}(t + 3)e^{3t} \cdot (-4e^{-3t}) + \frac{7}{5}(2t + 1)e^{-2t} \cdot e^{2t} \end{cases}$$

which simplifies to

$$\begin{cases} x_p = 3t + 3 \\ y_p = 2t - 1. \end{cases}$$

9.4 Nonlinear System

1. Differentiate the first equation to obtain

$$x'' = x'(a - by) - bxy'.$$

Use the second equation to eliminate y' :

$$\begin{aligned} x'' &= x'(a - by) - bx[-y(c - gx)] \\ &= ax' + (bcx - bx' - bgx^2)y. \end{aligned}$$

Now use the first equation to eliminate the remaining y.

$$x'' = ax' + b(cx - x' - gx^2) \cdot \frac{ax - x'}{bx}$$

This can also be expressed in the form

$$xx'' = (x')^2 + (gx^2 - cx)x' + acx^2 - agx^3.$$

Chapter 10

The Nonlinear Theory

10.1 Some Motivating Examples

10.2 Specializing Down

1. Derive equation (2).

Let s be the distance from the bob to its equilibrium position, measured along the path of motion (the circle). The force on the bob in the direction of motion is $F_g = -mg \sin x$. The damping force is $F_d = -cs'$ so, according to Newton's Second Law, $ms'' = -mg \sin x - cs'$. Using the fact that $s = ax$ this becomes $max'' = -mg \sin x - cax'$. Therefore,

$$x'' + \frac{c}{m}x' + \frac{g}{a}\sin x = 0.$$

3. The same curves, traversed in positive directions.

5. Let $x' = y$. Equation (1) is equivalent to

$$\begin{cases} x' = y \\ y' = -\frac{g}{a}\sin x. \end{cases}$$

The critical points are $(n\pi, 0)$, n an integer.

Equation (2) is equivalent to

$$\begin{cases} x' = y \\ y' = -\frac{g}{a}\sin x - \frac{c}{m}y. \end{cases}$$

It has the same critical points as Equation (1), $(n\pi, 0)$, n an integer. Equation (3) is equivalent to

$$\begin{cases} x' = y \\ y' = -x - \mu(x^2 - 1)y. \end{cases}$$

It has one critical point, $(0, 0)$.

7. Clearly $x = x_0 e^t$. Substitute this into the second equation, $y' = x_0 e^t + e^t$, to see that $y = x_0 e^t + e^t + A$. The constant A can be expressed in terms of $y(0) = y_0 : A = y_0 - x_0 - 1$, so the solution starting at (x_0, y_0) at $t = 0$ is

$$\begin{cases} x(t) = x_0 e^t \\ y(t) = (x_0 + 1)e^t + y_0 - x_0 - 1. \end{cases}$$

The trajectories are straight lines. Some of them are sketched below. The vertical trajectory starts at $(0, -3)$ and the horizontal trajectories all start at a point of the form $(-1, k)$. There are no equilibrium points.

10.3 Types of Critical Points: Stability

1. (a) All points $(x, 0)$ are critical points. None are isolated. Eliminating t, $\frac{dy}{dx} = \frac{2xy^2}{y(x^2+1)}$ so $\frac{dy}{y} = \frac{2xdx}{x^2+1}$. Consequently, $\ln y = \ln(x^2 + 1) + C$ and $y = C(x^2 + 1)$. The paths are parabolas. If $y > 0$ x is increasing and if $y < 0$, x is decreasing.

(b) The origin $(0, 0)$ is the only critical point. Eliminating t, $dy/dx = -x/y$ so $ydy = -xdx$. Consequently, $y^/2 = -x^2/2 + C$ and $y^2 + x^2 = C$. The paths are circles. If $y > 0$, then x is increasing so the circles are traversed in the clockwise direction.

(c) There are no critical points. Eliminating t, $dy/dx = \cos x$ so $y = \sin x + C$. The paths are sine waves. Since $x' > 0$ the tractories are traversed from left to right.

(d) All points $(0, y)$ are critical points. None are isolated. Eliminating t, $dy/dx = -2xy^2$ so $-dy/y^2 = 2xdx$ and $1/y = x^2 + C$. The paths lie on the curves $y = 1(x^2 + C)$. Since $x' > 0$ when $x < 0$ the trajectories in the left half-plane are traversed from left to right.

The tractories in the right half-plane are traversed from right to left. All points move toward the y-axis.

3. This second order equation is equivalent to the following system.

$$\begin{cases} x' = y \\ y' = 2x^3. \end{cases}$$

Therefore, the origin is an isolated critical point. Eliminating t, $dy/dx = 2x^3/y$, so $ydy = 2x^3dx$ and $y^2/2 = x^4/2 + C$. The solution trajectories lie on the curves $y^2 - x^4 = C$. If $y > 0$, then x is increasing; if $y < 0$, then x is decreasing. Several solution curves are sketched below. Note that the four trajectories that lie on the parabolas $y = \pm x^2$ not only form a natural boundary between two distinct classes of solution curves, but also seem to be attracting all trajectories to them as $t \to \infty$ and as $t \to -\infty$.

10.4 Critical Points and Stability for Linear Systems

1. (a) The auxiliary equation is $(m-2)(m-3) = 0$ so $m_1 = 2$ and $m_2 = 3$. The origin is an unstable node.

(b) THe auxiliary equation is $m^2 + 6m + 13 = 0$ with roots $m_{1,2} = -1 \pm 2i$. The origin is an asymptotically stable spiral.

(c) The auxiliary equation is $m^2 - 1 = 0$ with roots $m_1 = 1$ and $m_2 = -1$. The origin is an unstable saddle points.

(d) The auxiliary equation is $m^2 + 9 = 0$ with roots $\pm 3i$. The origin is a stable center.

(e) The auxiliary equation is $m^2+6m+9 = 0$ with roots $-3, -3$. The origin is an asymptotically stable borderline node.

(f) The auxiliary equation is $m^2 + 2m = 0$ with roots $m_1 = 0$ and $m_2 = -2$. The origin is not an isolated critical point. It can be classified as stable.

(g) The auxiliary equation is $m^2-6m+18 = 8$ with roots $m_{1,2} = 3\pm 3i$. The origin is an unstable spiral.

3. (a) The condition $a_1 b_1 - a_2 b_1 \neq 0$ guarantees that there is only one point (x_0, y_0) where $x' = 0$ and $y' = 0$ simultaneously.

(b) Make the substitution to obtain

$$\begin{cases} \frac{d\bar{x}}{dt} = a_1(\bar{x} + x_0) + b_1(\bar{y} + y_0) + c_1 \\ \frac{d\bar{y}}{dt} = a_2(\bar{x} + x_0) + b_2(\bar{y} + y_0) + c_2. \end{cases}$$

By the choice of x_0 and y_0 in part (a), this simplifies to

$$\begin{cases} \frac{d\bar{x}}{dt} = a_1\bar{x} + b_1\bar{y} \\ \frac{d\bar{y}}{dt} = a_2\bar{x} + b_2\bar{y}. \end{cases}$$

(c) The critical point is $(-3, 2)$–find the common zeros of the two linear terms on the right side. The eigenvalues for the system are negative: $m_{1,2} = -3 \pm \sqrt{3}$, so the critical point is an asymptotically stable node.

5. The hypotheses of Case E imply that $b_2 = -a_1$ and $a_1^2 + a_2 b_1 = -c^2$ for some positive number c. Begin by replacing b_2 with $-a_1$ and making the substitution $y/x = u$. This implies that $dy/dx = x du/dx + u$ and we have

$$x\frac{du}{dx} + u = \frac{a_2 - a_1 u}{a_1 + b_1 u}.$$

A little algebra yields

$$\frac{u + \frac{a_1}{b_1}}{u^2 + 2\frac{a_1}{b_1}u - \frac{a_2}{b_1}} du = -\frac{dx}{x}.$$

Complete the square in the denominator,

$$\frac{u + \frac{a_1}{b_1}}{u^2 + 2\frac{a_1}{b_1}u - \frac{a_2}{b_1}} du = -\frac{dx}{x},$$

and replace $a_1^2 + a_2 b_1$ with $-c^2$

$$\frac{u + \frac{a_1}{b_1}}{(u + \frac{a_1}{b_1})^2 + \frac{c^2}{b_1^2}} du = -\frac{dx}{x}.$$

This integrates to

$$\frac{1}{2}\ln\left[\left(u+\frac{a_1}{b_1}\right)^2+\frac{c^2}{b_1^2}\right]=-\ln x+A,$$

which, in terms of x and y, can be written in the form

$$\left(\frac{y}{x}+\frac{a_1}{b_1}\right)^2+\frac{c^2}{b_1^2}=\frac{B}{x^2},$$

where $B>0$. Multiply both sides by $b_1^2x^2$ to get $(b_1y+a_1x)^2+c^2x^2=b_1^2B$. Now expand the squared term and remove c in favor of $-a_1^2-a_2b_1$ to obtain

$$b_1^2y^2+2a_1b_1xy-a_2b_1x^2=b_1^2B^2.$$

This determines a one parameter family of ellipses because

$$4a_1^2b_1^2+4b_1^2a_2b_1=4b_1^2(a_1^2+a_2b_1)<0.$$

10.5 Stability by Liapunov's Direct Method

1. Each one can be classified using Theorem 11.5.4.

(a) Neither since $a>0$ and $b^2-4ac>0$.
(b) Of positive type since $a>0$ and $b^2-4ac<0$.
(c) Neither since $a<0$ and $b^2-4ac>0$.
(d) Of negative type because $a<0$ and $b^2-4ac<0$.

3. (a) As in Example 11.5.3, we seek a Liapunov function of the form $E(x,y)-ax^{2m}+by^{2n}$. OBserve that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F+\frac{\partial E}{\partial y}G&=2max^{2m-1}(-3x^3-y)+2nby^{2n-1}(x^5-2y^3)\\&=(2nby^{2n-1}x^5-2max^{2m-1}y)-(6max^{2m+2}+4nby^{2n+2}).\end{aligned}$$

We seek m and n (positive integers) and positive values for a and b that make the first term zero. Clearly $n=1$, $m=3$, $a=1$, and $b=3$ will do the job. Thus the function $E(x,y)=x^6+3y^2$ is of positive type and $(\partial E/\partial x)F+(\partial E/\partial y)G=-18x^8-12y^4$ is of negative type. According to Theorem 11.3, the origin is asymptotically stable.

(b) As above, consider $E(x,y) = ax^{2m} + by^{2n}$. Observe that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(-2x + xy^3) + 2nby^{2n-1}(x^2y^2 - y^3) \\ &= (2nbx^2y^{2n+1} + 2max^{2m}y^3) - (4max^{2m} + 2nby^{2n+2}).\end{aligned}$$

Let $n = 1, m = 1, a = 1$, and $b = 1$ and the function $E(x,y) = x^2 + y^2$ is of positive type while

$$\begin{aligned}(\partial E/\partial x)F + (\partial E/\partial y)G &= 4x^2y^3 - 4x^2 - 2y^4 \\ &= -4(1 - y^3)x^2 - 2y^4\end{aligned}$$

is of negative type when $|y| < 1$. The origin is asymptotically stable.

5. Consider $E(x,y) = ax^{2m} + by^{2n}$. Observe that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(2xy + x^3) + 2nby^{2n-1}(-x^2 + y^5) \\ &= (4max^{2m}y - 2nbx^2y^{2n-1}) + (2max^{2m+2} + 2nby^{2n+4}).\end{aligned}$$

Let $n = 1, m = 1, a = 1$, and $b = 2$ and the function $E(x,y) = x^2 + 2y^2$ is of positive type and so is $(\partial E/\partial x)F + (\partial E/\partial y)G = 2x^4 + 4y^6$. According to Exercise 4, the origin is unstable.

10.6 Simple Critical Points of Nonlinear Systems

1. $E(x,y) = x^2 + y^2$ is a Liapunov function for the first system with $(\partial E/\partial x)F + (\partial E/\partial y)G = -2x^4 - 2y^4$. Thus $(0,0)$ is an asymptotically stable critical point.

When applied to the second system, $E(x,y) = x^2 + y^2$ has the property that $(\partial E/\partial x)F + (\partial E/\partial y)G = 2x^4 + 2y^4$ implying that $(0,0)$ is unstable. See Exercise 4 in Section 11.5.

3. (a) Observe that $(0,0)$ is an isolated singular point of the linearized system. Moreover, when polar coordinates are used, $|2xy|/\sqrt{x^2 + y^2} \leq$

$2r$ and $3y^2/\sqrt{x^2+y^2} \le 3r$. Both approach 0 as $r \to 0$ so $(0,0)$ is a simple critical points. The auxiliary equation of the linearized system is $m^2 - 2m + 3 = 0$ with roots $m_{1,2} = 1 \pm \sqrt{2}i$. Thus $(0,0)$ is an unstable spiral for the linearized system and for the original system as well.

(b) Using polar coordinates, $|3x^2y|/\sqrt{x^2+y^2} \le 3r^2$ and, since $|\sin x| \le |x|$, $|y \sin x|/\sqrt{x^2+y^2} \le r$. Both approach 0 as $r \to 0$. Moreover, the linearized system has an isolated critical point at $(0,0)$ so the origin is a simple critical point. The auxiliary equation for the linearized system is $m^2 + 5m + 2 = 0$ with roots $m_{1,2} = \frac{-5 \pm \sqrt{17}}{2}$. Both roots are negative so the origin is an asymptotically stable node for the linearized system and for the original system as well.

5. Because $\det \begin{pmatrix} a_1 & b_1 \\ a)2 & b_2 \end{pmatrix} \ne 0$, $T(x,y) = \sqrt{(a_1x + b_1y)^2 + (a_2x + b_2y)^2}$ attains a positive minimum value m on the circle C of radius 1 centered at the origin. Suppose that there are nonzero critical points for the system (x_n, y_n) such that $(x_n, y_n) \to (0,0)$ as $n \to \infty$. Let $r_n = \sqrt{x_n^2 + y_n^2}$ and we have the following contradiction

$$\begin{aligned} 0 < m &\le T(x_n/r_n, y_n/r_n) \\ &= \frac{\sqrt{(a_1x_n + b_1y_n)^2 + (a_2x_n + b_2y_n)^2}}{r_n} \\ &= \frac{\sqrt{f(x_n,y_n)^2 + g(x_n,y_n)^2}}{r_n} \\ &\le \frac{|f(x_n,y_n)|}{r_n} + \frac{|g(x_n,y_n)|}{r_n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

10.7 Nonlinear Mechanics: Conservative Systems

1. The nonlinear spring equation is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -kx - \alpha x^3. \end{cases}$$

The solution curves satisfy the equation $dy/dx = -(kx + \alpha x^3)/y$ so $ydy = (-kx - \alpha x^3)dx$ and

$$y^2 = 2(-kx^2/2 - \alpha x^4/4) + E.$$

The solution trajectories lie on the curves $y = \pm\sqrt{E - (kx^2 + \alpha x^4/2)}$. The following pictures display the integral curves for $k = 1, \alpha = 1$ and for $k = 1, \alpha = -1$. **

3. For the hard spring in Exercise 1, $z = V(x) = x^2 + x^4/2$ and $V'(x) = 0$ only when $x = 0$; this is a minimum yielding a stable center at $(0, 0)$ as shown in the picture on the left.

In the picture on the right in Exercise 1, the spring is soft. $z = V(x) = x^2 - x^4/2$ and $V'(x) = 2x(1 - x^2)$ which is 0 at $x = 0$ and at $x = \pm 1$. The former is a local minimum for V–$(0, 0)$ is a stable center–and the latter two are absolute maxima yielding unstable saddle points at $(1, 0)$ and $(-1, 0)$ in the (x, y)-phase plane.

The phase plane trajectories and the energy curve for an inflection point are shown below. The critical point $(1, 0)$ is an unstable cusp.

10.8 The Poincaré-Bendixson Theorem

1. Introduce polar coordinates as was done for system (3) in this section. This yields

$$\begin{cases} \frac{dr}{dt} = r(3 - e^{r^2}) \\ \frac{d\theta}{dt} = 1. \end{cases}$$

Clearly there are periodic solutions: $r(t) = \sqrt{\ln 3}$, $\theta(t) = t + t_0$. In terms of x and y,

$$\begin{cases} x(t) = \sqrt{\ln 3}\cos(t + t_0) \\ y(t) = \sqrt{\ln 3}\sin(t + t_0). \end{cases}$$

3. Rescale the variable t with the substitution $\tau = \alpha t, \alpha > 0$, to obtain

$$a\alpha^2\frac{d^2x}{d\tau^2} + b\alpha(x^2 - 1)\frac{dx}{d\tau} + cx = 0.$$

Divide this equation by $a\alpha^2$ and let $\alpha = \sqrt{c/a}$ to get

$$\frac{d^2x}{d\tau^2} + \mu(x^2-1)\frac{dx}{d\tau} + x = 0,$$

where $\mu = b/\sqrt{ac}$.

10.9 Some Motivating Examples

10.10 Specializing Down

1. Derive equation (2).

Let s be the distance from the bob to its equilibrium position, measured along the path of motion (the circle). The force on the bob in the direction of motion is $F_g = -mg\sin x$. The damping force is $F_d = -cs'$ so, according to Newton's Second Law, $ms'' = -mg\sin x - cs'$. Using the fact that $s = ax$ this becomes $max'' = -mg\sin x - cax'$. Therefore,

$$x'' + \frac{c}{m}x' + \frac{g}{a}\sin x = 0.$$

3. The same curves, traversed in positive directions.

5. Let $x' = y$. Equation (1) is equivalent to

$$\begin{cases} x' = y \\ y' = -\frac{g}{a}\sin x. \end{cases}$$

The critical points are $(n\pi, 0)$, n an integer.

Equation (2) is equivalent to

$$\begin{cases} x' = y \\ y' = -\frac{g}{a}\sin x - \frac{c}{m}y. \end{cases}$$

It has the same critical points as Equation (1), $(n\pi, 0)$, n an integer.

Equation (3) is equivalent to

$$\begin{cases} x' = y \\ y' = -x - \mu(x^2-1)y. \end{cases}$$

It has one critical point, $(0, 0)$.

7. Clearly $x = x_0 e^t$. Substitute this into the second equation, $y' = x_0 e^t + e^t$, to see that $y = x_0 e^t + e^t + A$. The constant A can be expressed in terms of $y(0) = y_0 : A = y_0 - x_0 - 1$, so the solution starting at (x_0, y_0) at $t = 0$ is

$$\begin{cases} x(t) = x_0 e^t \\ y(t) = (x_0 + 1)e^t + y_0 - x_0 - 1. \end{cases}$$

The trajectories are straight lines. Some of them are sketched below. The vertical trajectory starts at $(0, -3)$ and the horizontal trajectories all start at a point of the form $(-1, k)$. There are no equilibrium points.

10.11 Types of Critical Points: Stability

1. (a) All points $(x, 0)$ are critical points. None are isolated. Eliminating t, $\frac{dy}{dx} = \frac{2xy^2}{y(x^2+1)}$ so $\frac{dy}{y} = \frac{2xdx}{x^2+1}$. Consequently, $\ln y = \ln(x^2 + 1) + C$ and $y = C(x^2 + 1)$. The paths are parabolas. If $y > 0$ x is increasing and if $y < 0$, x is decreasing.

(b) The origin $(0, 0)$ is the only critical point. Eliminating t, $dy/dx = -x/y$ so $ydy = -xdx$. Consequently, $y/2 = -x^2/2 + C$ and $y^2 + x^2 = C$. The paths are circles. If $y > 0$, then x is increasing so the circles are traversed in the clockwise direction.

(c) There are no critical points. Eliminating t, $dy/dx = \cos x$ so $y = \sin x + C$. The paths are sine waves. Since $x' > 0$ the tractories are traversed from left to right.

(d) All points $(0, y)$ are critical points. None are isolated. Eliminating t, $dy/dx = -2xy^2$ so $-dy/y^2 = 2xdx$ and $1/y = x^2+C$. The paths lie on the curves $y = 1(x^2 + C)$. Since $x' > 0$ when $x < 0$ the trajectories in the left half-plane are traversed from left to right. The tractories in the right half-plane are traversed from right to left. All points move toward the y-axis.

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3. This second order equation is equivalent to the following system.

$$\begin{cases} x' = y \\ y' = 2x^3. \end{cases}$$

Therefore, the origin is an isolated critical point. Eliminating t, $dy/dx = 2x^3/y$, so $ydy = 2x^3dx$ and $y^2/2 = x^4/2 + C$. The solution trajectories lie on the curves $y^2 - x^4 = C$. If $y > 0$, then x is increasing; if $y < 0$, then x is decreasing. Several solution curves are sketched below. Note that the four trajectories that lie on the parabolas $y = \pm x^2$ not only form a natural boundary between two distinct classes of solution curves, but also seem to be attracting all trajectories to them as $t \to \infty$ and as $t \to -\infty$.

10.12 Critical Points and Stability for Linear Systems

1. (a) The auxiliary equation is $(m-2)(m-3) = 0$ so $m_1 = 2$ and $m_2 = 3$. The origin is an unstable node.

(b) THe auxiliary equation is $m^2 + 6m + 13 = 0$ with roots $m_{1,2} = -1 \pm 2i$. The origin is an asymptotically stable spiral.

(c) The auxiliary equation is $m^2 - 1 = 0$ with roots $m_1 = 1$ and $m_2 = -1$. The origin is an unstable saddle points.

(d) The auxiliary equation is $m^2 + 9 = 0$ with roots $\pm 3i$. The origin is a stable center.

(e) The auxiliary equation is $m^2+6m+9 = 0$ with roots $-3, -3$. The origin is an asymptotically stable borderline node.

(f) The auxiliary equation is $m^2 + 2m = 0$ with roots $m_1 = 0$ and $m_2 = -2$. The origin is not an isolated critical point. It can be classified as stable.

(g) The auxiliary equation is $m^2-6m+18 = 8$ with roots $m_{1,2} = 3\pm 3i$. The origin is an unstable spiral.

3. (a) The condition $a_1b_1 - a_2b_1 \neq 0$ guarantees that there is only one point (x_0, y_0) where $x' = 0$ and $y' = 0$ simultaneously.

(b) Make the substitution to obtain

$$\begin{cases} \frac{d\bar{x}}{dt} = a_1(\bar{x} + x_0) + b_1(\bar{y} + y_0) + c_1 \\ \frac{d\bar{y}}{dt} = a_2(\bar{x} + x_0) + b_2(\bar{y} + y_0) + c_2. \end{cases}$$

By the choice of x_0 and y_0 in part (a), this simplifies to

$$\begin{cases} \frac{d\bar{x}}{dt} = a_1\bar{x} + b_1\bar{y} \\ \frac{d\bar{y}}{dt} = a_2\bar{x} + b_2\bar{y}. \end{cases}$$

(c) The critical point is $(-3, 2)$–find the common zeros of the two linear terms on the right side. The eigenvalues for the system are negative: $m_{1,2} = -3\pm\sqrt{3}$, so the critical point is an asymptotically stable node.

5. The hypotheses of Case E imply that $b_2 = -a_1$ and $a_1^2 + a_2b_1 = -c^2$ for some positive number c. Begin by replacing b_2 with $-a_1$ and making the substitution $y/x = u$. This implies that $dy/dx = xdu/dx + u$ and we have

$$x\frac{du}{dx} + u = \frac{a_2 - a_1u}{a_1 + b_1u}.$$

A little algebra yields

$$\frac{u + \frac{a_1}{b_1}}{u^2 + 2\frac{a_1}{b_1}u - \frac{a_2}{b_1}}du = -\frac{dx}{x}.$$

Complete the square in the denominator,

$$\frac{u + \frac{a_1}{b_1}}{u^2 + 2\frac{a_1}{b_1}u - \frac{a_2}{b_1}}du = -\frac{dx}{x},$$

and replace $a_1^2 + a_2b_1$ with $-c^2$

$$\frac{u + \frac{a_1}{b_1}}{(u + \frac{a_1}{b_1})^2 + \frac{c^2}{b_1^2}}du = -\frac{dx}{x}.$$

This integrates to

$$\frac{1}{2}\ln\left[\left(u + \frac{a_1}{b_1}\right)^2 + \frac{c^2}{b_1^2}\right] = -\ln x + A,$$

which, in terms of x and y, can be written in the form

$$\left(\frac{y}{x} + \frac{a_1}{b_1}\right)^2 + \frac{c^2}{b_1^2} = \frac{B}{x^2},$$

where $B > 0$. Multiply both sides by $b_1^2x^2$ to get $(b_1y + a_1x)^2 + c^2x^2 = b_1^2B$. Now expand the squared term and remove c in favor of $-a_1^2 - a_2b_1$ to obtain

$$b_1^2y^2 + 2a_1b_1xy - a_2b_1x^2 = b_1^2B^2.$$

This determines a one parameter family of ellipses because

$$4a_1^2b_1^2 + 4b_1^2a_2b_1 = 4b_1^2(a_1^2 + a_2b_1) < 0.$$

10.13 Stability by Liapunov's Direct Method

1. Each one can be classified using Theorem 11.5.4.

(a) Neither since $a > 0$ and $b^2 - 4ac > 0$.

(b) Of positive type since $a > 0$ and $b^2 - 4ac < 0$.

(c) Neither since $a < 0$ and $b^2 - 4ac > 0$.

(d) Of negative type because $a < 0$ and $b^2 - 4ac < 0$.

3. (a) As in Example 11.5.3, we seek a Liapunov function of the form $E(x, y) - ax^{2m} + by^{2n}$. OBserve that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(-3x^3 - y) + 2nby^{2n-1}(x^5 - 2y^3)\\ &= (2nby^{2n-1}x^5 - 2max^{2m-1}y) - (6max^{2m+2} + 4nby^{2n+2}).\end{aligned}$$

We seek m and n (positive integers) and positive values for a and b that make the first term zero. Clearly $n = 1$, $m = 3$, $a = 1$, and $b = 3$ will do the job. Thus the function $E(x, y) = x^6 + 3y^2$ is of positive type and $(\partial E/\partial x)F + (\partial E/\partial y)G = -18x^8 - 12y^4$ is of negative type. According to Theorem 11.3, the origin is asymptotically stable.

(b) As above, consider $E(x, y) = ax^{2m} + by^{2n}$. Observe that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(-2x + xy^3) + 2nby^{2n-1}(x^2y^2 - y^3)\\ &= (2nbx^2y^{2n+1} + 2max^{2m}y^3) - (4max^{2m} + 2nby^{2n+2}).\end{aligned}$$

Let $n = 1, m = 1, a = 1$, and $b = 1$ and the function $E(x, y) = x^2 + y^2$ is of positive type while

$$\begin{aligned}(\partial E/\partial x)F + (\partial E/\partial y)G &= 4x^2y^3 - 4x^2 - 2y^4 \\ &= -4(1 - y^3)x^2 - 2y^4\end{aligned}$$

is of negative type when $|y| < 1$. The origin is asymptotically stable.

5. Consider $E(x, y) = ax^{2m} + by^{2n}$. Observe that for such a function

$$\begin{aligned}\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G &= 2max^{2m-1}(2xy + x^3) + 2nby^{2n-1}(-x^2 + y^5) \\ &= (4max^{2m}y - 2nbx^2y^{2n-1}) + (2max^{2m+2} + 2nby^{2n+4}).\end{aligned}$$

Let $n = 1, m = 1, a = 1$, and $b = 2$ and the function $E(x, y) = x^2+2y^2$ is of positive type and so is $(\partial E/\partial x)F+(\partial E/\partial y)G = 2x^4+4y^6$. According to Exercise 4, the origin is unstable.

10.14 Simple Critical Points of Nonlinear Systems

1. $E(x, y) = x^2 + y^2$ is a Liapunov function for the first system with $(\partial E/\partial x)F+(\partial E/\partial y)G = -2x^4-2y^4$. Thus $(0, 0)$ is an asymptotically stable critical point.

When applied to the second system, $E(x, y) = x^2+y^2$ has the property that $(\partial E/\partial x)F+(\partial E/\partial y)G = 2x^4+2y^4$ implying that $(0, 0)$ is unstable. See Exercise 4 in Section 11.5.

3. (a) Observe that $(0, 0)$ is an isolated singular point of the linearized system. Moreover, when polar coordinates are used, $|2xy|/\sqrt{x^2 + y^2} \leq 2r$ and $3y^2/\sqrt{x^2 + y^2} \leq 3r$. Both approach 0 as $r \to 0$ so $(0, 0)$ is a simple critical points. The auxiliary equation of the linearized system is $m^2 - 2m + 3 = 0$ with roots $m_{1,2} = 1 \pm \sqrt{2}i$. Thus $(0, 0)$ is an unstable spiral for the linearized system and for the original system as well.

(b) Using polar coordinates, $|3x^2y|/\sqrt{x^2+y^2} \le 3r^2$ and, since $|\sin x| \le |x|$, $|y \sin x|/\sqrt{x^2+y^2} \le r$. Both approach 0 as $r \to 0$. Moreover, the linearized system has an isolated critical point at $(0,0)$ so the origin is a simple critical point. The auxiliary equation for the linearized system is $m^2 + 5m + 2 = 0$ with roots $m_{1,2} = \frac{-5\pm\sqrt{17}}{2}$. Both roots are negative so the origin is an asymptotically stable node for the linearized system and for the original system as well.

5. Because $\det \begin{pmatrix} a_1 & b_1 \\ a)2 & b_2 \end{pmatrix} \ne 0$, $T(x,y) = \sqrt{(a_1x+b_1y)^2+(a_2x+b_2y)^2}$ attains a positive minimum value m on the circle C of radius 1 centered at the origin. Suppose that there are nonzero critical points for the system (x_n, y_n) such that $(x_n, y_n) \to (0,0)$ as $n \to \infty$. Let $r_n = \sqrt{x_n^2 + y_n^2}$ and we have the following contradiction

$$\begin{aligned} 0 < m &\le T(x_n/r_n, y_n/r_n) \\ &= \frac{\sqrt{(a_1x_n+b_1y_n)^2+(a_2x_n+b_2y_n)^2}}{r_n} \\ &= \frac{\sqrt{f(x_n,y_n)^2+g(x_n,y_n)^2}}{r_n} \\ &\le \frac{|f(x_n,y_n)|}{r_n} + \frac{|g(x_n,y_n)|}{r_n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

10.15 Nonlinear Mechanics: Conservative Systems

1. The nonlinear spring equation is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -kx - \alpha x^3. \end{cases}$$

The solution curves satisfy the equation $dy/dx = -(kx+\alpha x^3)/y$ so $ydy = (-kx - \alpha x^3)dx$ and

$$y^2 = 2(-kx^2/2 - \alpha x^4/4) + E.$$

The solution trajectories lie on the curves $y = \pm\sqrt{E - (kx^2 + \alpha x^4/2)}$. The following pictures display the integral curves for $k = 1, \alpha = 1$ and for $k = 1, \alpha = -1$. **

3. For the hard spring in Exercise 1, $z = V(x) = x^2 + x^4/2$ and $V'(x) = 0$ only when $x = 0$; this is a minimum yielding a stable center at $(0, 0)$ as shown in the picture on the left.

In the picture on the right in Exercise 1, the spring is soft. $z = V(x) = x^2 - x^4/2$ and $V'(x) = 2x(1 - x^2)$ which is 0 at $x = 0$ and at $x = \pm 1$. The former is a local minimum for V–$(0, 0)$ is a stable center–and the latter two are absolute maxima yielding unstable saddle points at $(1, 0)$ and $(-1, 0)$ in the (x, y)-phase plane.

The phase plane trajectories and the energy curve for an inflection point are shown below. The critical point $(1, 0)$ is an unstable cusp.

10.16 The Poincaré-Bendixson Theorem

1. Introduce polar coordinates as was done for system (3) in this section. This yields

$$\begin{cases} \frac{dr}{dt} = r(3 - e^{r^2}) \\ \frac{d\theta}{dt} = 1. \end{cases}$$

Clearly there are periodic solutions: $r(t) = \sqrt{\ln 3}$, $\theta(t) = t + t_0$. In terms of x and y,

$$\begin{cases} x(t) = \sqrt{\ln 3}\cos(t + t_0) \\ y(t) = \sqrt{\ln 3}\sin(t + t_0). \end{cases}$$

3. Rescale the variable t with the substitution $\tau = \alpha t, \alpha > 0$, to obtain

$$a\alpha^2 \frac{d^2x}{d\tau^2} + b\alpha(x^2 - 1)\frac{dx}{d\tau} + cx = 0.$$

Divide this equation by $a\alpha^2$ and let $\alpha = \sqrt{c/a}$ to get

$$\frac{d^2x}{d\tau^2} + \mu(x^2 - 1)\frac{dx}{d\tau} + x = 0,$$

where $\mu = b/\sqrt{ac}$.